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Abstract We consider the problem of *explicitly* pricing and hedging an option written on a non-exchangeable asset when trading in a correlated asset is possible. This is a typical case of incomplete market where it is well known that the super-replication concept provides generally too high prices. Here, following J.H. Cochrane and J. Saá-Requejo, we study valuation under No Good Deal (NGD) Assumption. First, we clarify the notion of NGD. Then we compute a lower and an upper bound for NGD price and show numerically that it can be significatively higher that the one previously compute in the literature. We then propose several hedging strategies starting from NGD price and show numerically that the minimum variance one is quite efficient.

Keywords No Good Deal · basis risk · mean variance hedging
JEL Classification Numbers C61 · C63 · G11 · G13

1 Introduction

In this paper, we provide new and concrete elements for pricing and hedging Basis Risk. We consider the problem of an agent paying a derivative written on a risky asset V on which trading is not possible, not allowed or too costly. For example, an investor can sell an option on a stock and prefer for liquidity reasons to hedge with an associated index; or in the commodities market hedge with Fioul Oil 1% an option on Fioul Oil Straight Run 0,5%. In these cases, one considers a more liquid asset S which is highly correlated with the underlying V and then price and hedge investing in S and cash only.

This is a typical incomplete market and the natural extension of No Arbitrage pricing, i.e. replication, is the super-replication concept. But, in the Black-Scholes diffusion world, it is well known that this lead to unreasonably high valuations. For example, the super-replication price of a call option on a non-tradable asset is equal to the initial value of this asset, provided that it is possible to buy it at the beginning of the trading period.

An alternative method has been introduced by (Cochrane and Saa-Requejo, 2001): the No Good Deal (NGD) pricing. The idea is to exclude from admissible strategies, portfolios which have too high "Sharpe Ratio" because, similarly to arbitrage opportunities, good deals would quickly disappear as investors would immediately grab them. But how should we define the Sharpe ratio? In economic theory, the Sharpe ratio of a claim measures the degree to which the expected return of the claim exceeds the risk free rate, as a proportion of the standard deviation of this claim. For

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dynamic strategy, the meaning of Sharpe ratio is not so clear and there exist different definitions in the literature. We refer to Cochrane and Saa-Requejo (2001), Björk and Slinko (2006), Bayraktar and Young (2008) or Klöppel and Schweizer (2007) among others.

Klöppel and Schweizer (2007) define Sharpe Ratio globally and find that this NGD constraint, i.e. imposing a bound on the Sharpe ratio of any portfolio based on exchangeable claims, is equivalent to a bound on the variance of the density of the pricing measures. Note that this definition of Sharpe Ratio and No Good Deal price is linked to the notion of coherent risk measure and coherent NGD utility function of Cherny (2008).

Cochrane and Saa-Requejo (2001) and Björk and Slinko (2006) use an instantaneous notion of Sharpe Ratio and the authors assert that the NGD constraint leads to a bound on the market risk premium (considering both coverable and uncoverable risks). We remark that only a bound on the coverable risk premium naturally appears and consequently, it seems that their notion of NGD price is not directly related to instantaneous definition of Sharpe Ratio. We also show that it is also not related to the global Sharpe Ratio.

We choose to define No Good Deal using a global Sharpe Ratio similar to the one of Klöppel and Schweizer (2007). Then we introduce NGD price as the minimum initial wealth such that there exists a strategy leading to a residual wealth (after delivering the claim) having a positive coherent NGD utility function (see (16) and (17)). As the super-replication price, the NGD price can be dually represented by the supremum over all pricing measures with a bounded variance by some constant related to the NGD constraint. The pricing measure (also called equivalent martingale measure or EMM) can be represented by their densities which depend on the coverable and uncoverable risk premium. This last quantity is a stochastic process, and it is not possible to transfer our maximization constraint on it. In fact, if we set a bound on the market (coverable and uncoverable) risk premium, then the global Sharpe Ratio is bounded but the reverse is not true. Thus, we expect to find that the NGD price is higher that the one previously compute by Cochrane and Saa-Requejo (2001). The optimisation problem associated to NGD constraint is difficult to solve : we propose explicit upper and lower bounds for it and provide an analytical recipe to compute them. Then we show that our lower bound can be significatively higher than the prices computed by Cochrane and Saa-Requejo (2001) or Björk and Slinko (2006) (which are equal).

We then turn to the hedging issues. We first point out that in contrast to the super-replication notion, no natural strategy appears from our No Good Deal concept. So we compare several hedging strategies. First, we introduce very basic one as buy and hold and Black Scholes related to some approximated option (see (46)). Then, we discuss the hedging criterium introduced by Becherer (2009), which is, in our context, related to the price introduced by Cochrane and Saa-Requejo (2001). His hedging strategy is such that the NGD price of Cochrane and Saa-Requejo (2001) is equal to the minimum, over all admissible strategies, of some adhoc risk measure of the replication error (see (47)). First, we show that his hedging criterium does not apply to our NGD price : the construction of an ad hoc risk measure in our NGD context is left to further study. From our point of view, the drawback of Becherer (2009) criterium is that it is not explicit (it could be obtained as the numerical solution of a BSDE) and thus economically not easy to understand. So we propose a quadratic criterium (but different from the NGD one), for which explicit and fully calculable results are possible : to minimize the variance of the replication error under the historical probability. This notion has been first introduced by Duffie and Richardson (1991) and Schweizer (1992) and extended by Gourieroux et al. (1998), and it leads to hedging and pricing corresponding to Minimum Variance criterium. It is a quadratic minimization problem. The idea is to project our derivative product on the set of all admissible payoffs induced by the tradable asset S and the cash. Since those assets are not martingales under the historical probability, this is not technically possible and therefore we will use the classical tool of change of numéraire. Our contributions are the following : we propose a streamline proof with a direct derivation of the "right" numéraire (without using the so called minimum variance measure) and solve explicitly the minimum variance problem in cases of Basis Risk. We compute a closed form formula for both the hedging strategies and the error associated to these strategies. This error can be divided in two parts.

Then we perform numerical experiments. We consider a non-exchangeable asset which is more risky but provides higher returns than the exchangeable one. We are typically in the case of a very liquid index and a less liquid constituent of this index. We compute the finale value of the different hedging strategies and compare them through three points of view : probability of super-replication, expected loss and VaR. We obtain that the strategy of Becherer (2009) and the minimum variance one have very similar results and that they perform better than the others strategies. Thus if we want to obtain fully calculable and easy to interpret results, minimum variance strategy might be a satisfying hedging criterium.

The rest of the paper is organized as follows: in section 2, we present the financial model. We define the set of EMM, which must be in L^2 in our context. We relate the coverable and uncoverable risk premium to the variance of densities of EMM. In section 3, we review the various notions of "Sharpe Ratio" in the literature and their implication for No Good Deal price definition. In section 4, we discuss different prices for basis risk and provide some comparison results. Section 5 deals with hedging strategy issues. The technical proofs of the paper are grouped in the Appendix.

2 The financial model

We consider the problem of pricing and hedging a derivative product written on a risky asset V, on which trading is not possible. We assume that we can observe the price of V at all times. We will investigate the case where there exists a risky asset S, which is similar to V and traded in the market. This similarity will be measured by the correlation between the respective risk sources of the two assets. The financial market contains also a non-risky asset called S^0.
Let $(W_t)_{0\le t\le T}$ and $(W^*_t)_{0\le t\le T}$ be two independent real-valued Brownian motion, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that,

$$
\begin{aligned}
dS^0_t &= S^0_t r dt &(1)\\
dS_t &= S_t(\mu_S dt + \sigma_S dW_t) &(2)\\
dV_t &= V_t(\mu_V dt + \sigma_V(\rho dW_t + \sqrt{1-\rho^2} dW^*_t)) &(3)
\end{aligned}
$$

where r, μ_S, σ_S, μ_V and σ_V are $\mathbb{R}$-valued coefficients representing respectively the instantaneous risk free rate, the drift and volatility of S and the ones of V. ρ is the correlation between risk sources of the two assets (W and $\rho W + \sqrt{1-\rho^2}W^*$) with $-1 < \rho < 1$.

We will use the notations $h_S = \frac{\mu_S - r}{\sigma_S}$ and $h_V = \frac{\mu_V - r}{\sigma_V}$ for Sharpe ratio (in the classical sense) of the assets S and V respectively.

We introduce now some further useful notations. For any Brownian motion B, $\mathcal{F}^B = \{\mathcal{F}^B_t, 0 \le t \le T\}$ is the $\mathbb{P}$-augmentation of the filtration generated by B. We shall denote by $\mathcal{F}_t = \mathcal{F}^{W,W^*}_t$ for $0 \le t \le T$ and $\mathbb{F} = \mathcal{F}^{W,W^*}$ the flow of total information on $[0, T]$, where $T > 0$ is a finite time horizon.

For any probability $\mathbb{Q}$ on $(\Omega, \mathcal{F})$, $L^0(\mathbb{Q})$, $L^\infty(\mathbb{Q})$, $L^p(\mathbb{Q})$ for $p > 0$ will represent respectively the set of measurable, measurable and $\mathbb{Q}$-almost surely bounded, and measurable and such that the p-moment exists, random variables. For any Brownian motion B, $L^2_{loc}(B)$ will be the space of $\mathcal{F}^B$ progressively measurable processes λ such that $\int_0^T \lambda_s^2 ds < +\infty$. Finally, if X is a semi-martingale, $L(X)$ is the set of progressively measurable processes integrable with respect to X (see for example Protter (1990), p134).

The expectation and the variance computed under $\mathbb{P}$ will be denoted by $\mathbb{E}$ and Var, the expectation and the variance computed under $\mathbb{Q}$ will be denoted by $\mathbb{E}^{\mathbb{Q}}$ and $\mathrm{Var}^{\mathbb{Q}}$.
Let us introduce the set of pricing measure

$$\mathcal{M}^e(\mathbb{P}) = \left\{\mathbb{Q} \sim \mathbb{P} \;\; : \; S/S^0 \text{ is a } \mathbb{Q} \text{ martingale } \right\}.$$

The Fundamental Theorem of Asset Pricing asserts that $\mathcal{M}^e(\mathbb{P}) \neq \emptyset$ under some kind of no arbitrage condition, see Delbaen and Schachermayer (1994) [1]. In our simple setup, it is easy to see directly that $\mathcal{M}^e(\mathbb{P})$ is non-empty (see below) and thus that the no arbitrage condition holds.
In the context of pricing under No Good Deal principle, we need to introduce the space

$$\mathcal{M}^2(\mathbb{P}) \; := L^2(\mathbb{P}) \cap \mathcal{M}^e(\mathbb{P}).^2$$

In order to characterize $\mathcal{M}^2(\mathbb{P})$, we define Y_T^λ and Z_T^λ, for $\lambda \in L^2_{loc}((W, W^*))$, by

$$Y_T^\lambda := \exp\left(\int_0^T \lambda_s dW_s^\star - \frac{1}{2}\int_0^T \lambda_s^2 ds\right) \tag{4}$$

$$Z_T^\lambda := \exp\left(-h_S W_T - \frac{1}{2}h_S^2 T + \int_0^T \lambda_s dW_s^\star - \frac{1}{2}\int_0^T \lambda_s^2 ds\right) = Z_T^0 Y_T^\lambda \tag{5}$$

We also denote by Λ the set of $\lambda \in L^2_{loc}((W, W^*))$ such that Z_T^λ is a square integrable martingale. The number h_S is interpreted as the risk premium of the hedgeable risk W and $-\lambda$ as the risk premium of the non-hedgeable risk W^*. From now, we call $\mathbb{Q}^\lambda$ the probability measure such that $Z_T^\lambda = d\mathbb{Q}^\lambda/d\mathbb{P}$, for $\lambda \in \Lambda$. It is well known that $\mathcal{M}^2(\mathbb{P})$ (it is essentially Girsanov Theorem, see for example Musiela and Rutkowski (2007)) is explicitly given by

$$\mathcal{M}^2(\mathbb{P}) = \left\{\mathbb{Q} \mid \exists \lambda \in \Lambda \text{ s.t. } \frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T^\lambda\right\} \tag{6}$$

and is non-empty. Thus the measure $\mathbb{Q}^\lambda$ are the so called pricing measure. Note that in a market where only the information on the tradeable asset S is available (i.e. the filtration is $\mathcal{F}^W$), $\mathcal{M}^2(\mathbb{P}) = \{\mathbb{Q}^0\}$.
For the sequel, we need to express $\mathrm{Var}(Z_T^\lambda)$ for any $\lambda \in \Lambda$.

Lemma 1 *Let $\lambda \in \Lambda$, then*

$$\mathrm{Var}(Z_T^\lambda) = e^{h_S^2 T}\mathbb{E}^{\tilde{\mathbb{Q}}}\left(e^{\int_0^T \lambda_s^2 ds}\right) - 1, \tag{7}$$

for some $\tilde{\mathbb{Q}} \sim \mathbb{P}$, $\mathbb{Q} \in L^2(\mathbb{P})$. If λ is a constant process we get that

$$\mathrm{Var}(Z_T^\lambda) = e^{(h_S^2+\lambda^2)T} - 1. \tag{8}$$

Note that a bound on the process λ implies a bound on the L^2 moment of the density Z_T^λ. This remark will be fundamental when defining the Sharpe Ratio of a wealth process.

Proof See Appendix 6.1.2

We now define the space of trading strategies in (S^0, S) denoted by $\mathcal{S}$. Two kinds of constraints need to be impose on a strategy (Φ^0, Φ^1) : (i) conditions such that the associated wealth $X_t := \Phi_t^0 S_t^0 + \Phi_t^1 S_t$ is in $L^2(\mathbb{P})$ and (ii) conditions in order to avoid strategies leading to arbitrage.

Definition 1 *A strategy $(\Phi^0, \Phi^1) \in \mathcal{S}$ is a $\mathbb{R}^2$-valued predictable process such that :*
(i) the associated wealth process X is defined by $X_t := \Phi_t^0 S_t^0 + \Phi_t^1 S_t \in L^2(\mathbb{P})$
(ii) $\frac{X_t}{S_t^0}$ is a $\mathbb{Q}$-martingale under all $\mathbb{Q} \in \mathcal{M}^2(\mathbb{P})$.

Definition 2 *A strategy $(\Phi^0, \Phi^1) \in \mathcal{S}$ is self-financing in $\left(S^0, S\right)$ if and only if :*

$$\begin{aligned}(\Phi^0, \Phi^1) &\in L\left(S^0, S\right)\\ \Phi^0 dS^0 + \Phi^1 dS &= d\left(\Phi^0 S^0 + \Phi^1 S\right)\end{aligned} \tag{9}$$

The set of such strategies is called $\mathcal{A}_2$.

[1] In general this holds true for local martingale.
[2] by $\mathbb{Q} \in L^2(\mathbb{P})$, we mean that the density of $\mathbb{Q}$ w.r.t. $\mathbb{P}$ is in L^2.

It is easy to see that if $(\Phi^0, \Phi^1) \in \mathcal{A}_2$, then we get

$$X_t = S_t^0 \left(X_0 + \int_0^t \Phi_s^1 d\frac{S_s}{S_s^0} \right). \tag{10}$$

3 Defining a Good Deal

Roughly speaking, a good deal is an asset or a strategy with a Sharpe ratio too high. Similarly to arbitrage opportunities, good deals will quickly disappear as investors would use them in priority. The idea of Cochrane and Saa-Requejo (2001) is thus to exclude good deals as well as arbitrage opportunities. The question however is how to define the Sharpe ratio? In economic theory, the Sharpe ratio of a claim measures the degree to which the expected return of the claim exceeds the risk free rate, as a proportion of the standard deviation of this claim. To formalize this in an abstract setup, there exists several definitions in the literature. We first analyze them in our context and conclude which is the "right" definition to use.

The first definition, the so-called conditional instantaneous Sharpe Ratio can be found, for example, in Björk and Slinko (2006) or Bayraktar and Young (2008). Let X_t be the value of a self financing strategy at time t. The Sharpe ratio is defined, formally, by :

$$SR^1(X_t) = \frac{\frac{1}{dt}\mathbb{E}\left(\frac{dX_t}{X_t}/\mathcal{F}_t\right) - r}{\frac{1}{dt}\sqrt{\mathrm{Var}\left(\frac{dX_t}{X_t}/\mathcal{F}_t\right)}} \tag{11}$$

Note first, that the Sharpe ratio is not a number but a stochastic process. Clearly, the value of the Sharpe ratio will depend upon the type of strategies which are allowed. In Björk and Slinko (2006), only trading in the non-risky asset S^0 and the exchangeable asset S are allowed. Let X_t be the value at time t of a self-financing strategy $(\Phi^0, \Phi^1) \in \mathcal{A}_2 : X_t = \Phi_t^0 S_t^0 + \Phi_t^1 S_t$. Using the self-financing condition and Equations (1) and (2) :

$$dX_t = \Phi_t^0 dS_t^0 + \Phi_t^1 dS_t = X_t \left(r + \Phi_t^1 S_t \frac{\mu_S - r}{X_t} \right) dt + S_t \Phi_t^1 \sigma_S dW_t.$$

It follows that

$$\mathbb{E}\left(\frac{dX_t}{X_t}/\mathcal{F}_t\right) = \left(r + \Phi_t^1 S_t \frac{\mu_S - r}{X_t} \right) dt \text{ and } \sqrt{\mathrm{Var}\left(\frac{dX_t}{X_t}/\mathcal{F}_t\right)} = \frac{S_t \Phi_t^1 \sigma_S}{X_t} dt$$

Thus

$$SR^1(X_t) = \frac{\mu_S - r}{\sigma_S} = h_S.$$

This last quantity is the Sharpe ratio (in the classical sense) of the risky asset S and the conditional instantaneous Sharpe ratio (which is, in general, a stochastic process) reduces here to a number. Björk and Slinko (2006) argue that they consider the Sharpe ratio of the entire economy (see their Remark 3.4). As far as we understand, they only consider trading in the tradeable underlying securities (S^0 and S) and in derivatives which can be attained by trading in those underlying assets.

Bayraktar and Young (2008) also use the notion defined in (11) but they consider the Sharpe ratio of a portfolio consisting of the tradeable underlying assets and the derivative H they want to price. The difference is major since the price of H depends on the non-tradeable asset V. In a first time, they find the portfolio in the tradeable underlying securities that minimizes the local variance of the global portfolio (including the derivative). The price of the derivative is then obtained by fixing the instantaneous Sharpe ratio at some given value.

We now turn for the second kind of definition of Sharpe ratio, the so-called unconditional global Sharpe Ratio which can be found in Klöppel and Schweizer (2007). The unconditional global Sharpe ratio of a claim measures the degree to which the expected return of the claim exceeds the expected return computed under a risk neutral pricing measure, as a proportion of the standard deviation of this claim. The definition is formally given for any claim X on $(\Omega, \mathcal{F}, \mathbb{P})$ and depends on a measure $\mathbb{Q} \in \mathcal{M}^2(\mathbb{P})$.

$$SR^2(X, \mathbb{Q}) = \frac{\mathbb{E}(X) - \mathbb{E}^{\mathbb{Q}}(X)}{\sqrt{\operatorname{Var}(X)}} \tag{12}$$

If X is constant or $\operatorname{Var}(X) = \infty$, we set $SR^2(X, \mathbb{Q}) = 0$. Note that the Sharpe ratio will be well-defined if $X \in L^2(\mathbb{P})$ (as $\mathbb{Q} \in L^2(\mathbb{P})$, the Cauchy Schwarz inequality implies that $X \in L^1(\mathbb{Q})$). For $\mathbb{Q} \in \mathcal{M}^e(\mathbb{P})$ let

$$\mathcal{C}(\mathbb{Q}) = \{X \in L^0(\mathbb{P}) \ : X^- \in L^\infty(\mathbb{P}) \text{ and } \mathbb{E}^{\mathbb{Q}}(X) < \infty\}.$$

This set can be interpreted as the set of claims, which are bounded from below (in order to avoid doubling strategies) and such that their price under the pricing measure $\mathbb{Q}$ is finite and thus affordable from $\mathbb{E}^{\mathbb{Q}}[X]$ if we believe that the pricing measure is $\mathbb{Q}$. It is easy to see that if $X \in \mathcal{C}(\mathbb{Q}) \cap \{X \ : \mathbb{E}(X) < \infty\} \subset L^1(\mathbb{P})$ the Sharpe ratio is also well-defined.

It is clear from the definition that this second notion of Sharpe ratio is intimately linked to the choice of a pricing measure : if you believe the right pricing measure is $\mathbb{Q}$, the Sharpe ratio measures, in proportion of standard deviation, the excess between the expected value and the price. It is also a global measure of the performance of a claim X. Moreover it has the following remarkable property :

Proposition 1 *Let* $\mathbb{Q} \in \mathcal{M}^2(\mathbb{P})$ *then*

$$\sup_{X \in \mathcal{C}(\mathbb{Q}) \cap \{X \ : \mathbb{E}(X) < \infty\}} SR^2(X, \mathbb{Q}) = \sqrt{\operatorname{Var} Z_T}.$$

The proof can be found in the Phd Thesis of Klöppel (2006) but for the ease of reading, we report it below :

Proof First we prove that the supremum is less or equal to $\sqrt{\operatorname{Var} Z_T}$. If $X \in \mathcal{C}(\mathbb{Q}) \cap \{X \ : \mathbb{E}(X) < \infty\} \setminus L^2(\mathbb{P})$ then $SR^2(X, \mathbb{Q}) = 0$ else as $\mathbb{E}(X) - \mathbb{E}^{\mathbb{Q}}(X) = \mathbb{E}[(X - \mathbb{E}(X))(1 - Z_T)]$, the required inequality follows from Cauchy-Schwarz inequality. In order to prove that there is in fact equality, consider the sequence $X_n = \mathbb{E}^{\mathbb{Q}}[X] - Z_T \mathbf{1}_{Z_T \leq n}$, then $X_n \in \mathcal{C}(\mathbb{Q}) \cap \{X \ : \mathbb{E}(X) < \infty\}$. Moreover,

$$SR^2(X_n, \mathbb{Q}) = \frac{\mathbb{E}\left((Z_T \mathbf{1}_{Z_T \leq n}(Z_T - 1)\right)}{\sqrt{\operatorname{Var}(Z_T \mathbf{1}_{Z_T \leq n})}},$$

which converges by Lebesgue's Theorem to $\sqrt{\operatorname{Var}(Z_T)}$. Note that if the supremum is computed in $L^2(\mathbb{P})$, then it is a maximum.

So, for a given pricing measure, imposing a bound on the Sharpe ratio SR^2 of all affordable claims is equivalent to imposing exactly the same bound on the variance of the density of the pricing measure. As already mentioned in the introduction, pricing under the No Good Deal assumption requires to compute the supremum of the discounted claim under all the pricing measures when excluding the Good Deals, i.e. when putting a bound on the Sharpe Ratio of all affordable claim. With SR^2 definition, it means to compute the supremum of the discounted claim under all pricing measures with a bounded variance. The result of Proposition 1 is thus very important for the solution of our problem of pricing, since with the definition of SR^1, it is not possible to achieve the same conclusion : recall that $SR^1(X_t) = h_S$. The information obtained using a bound on SR^2 is thus richer than the one using SR^1. So, we will choose to define the Sharpe Ratio SR^2 (see equation (12)).

Below we clarify the restriction used for pricing under No Good Deal by Cochrane and Saa-Requejo (2001) and Björk and Slinko (2006) and explain why, in our opinion, it is not directly related to a restriction on the Sharpe Ratio defined by SR^1 or by SR^2. First, we recall that Cochrane and Saa-Requejo (2001) defined their No Good Deal pricing rule by imposing a bound on $\frac{1}{dt}\mathbb{E}\left[\left(\frac{dZ_t^\lambda}{Z_t^\lambda}\right)^2\right]$, which is equivalent to bound $(\lambda_t)_t$, the risk premium process on the non-coverable risk (recall Equation (5)). Björk and Slinko (2006) also defined their No Good Deal pricing rule by putting a bound on $(\lambda_t)_t$.

The first question is how to relate a restriction on $(\lambda_t)_t$ to a bound on SR^1 or SR^2. As far as we understand, the argument of Björk and Slinko (2006), following Hansen and Jagannathan (1991), is to say that $|SR^1(X_t)| = |h_S| \le |(-h_S, \lambda_t)|_{\mathbb{R}^2}$. Then, instead of imposing a bound on the Sharpe Ratio $SR^1(X_t)$, they rather put a bound on $|(-h_S, \lambda_t)|_{\mathbb{R}^2}$. This is of course mathematically correct but, in our opinion, it is not economically meaningful because from the first definition of Sharpe ratio, only a bound on h_S, the risk premium on the coverable risk W, naturally appears. Thus, imposing a bound on $(\lambda_t)_t$ is not economically related to the Sharpe Ratio definition SR^1.

The next question is then : is it mathematically equivalent to put a bound on SR^2 or on $(\lambda_t)_t$? The answer is no, except when $(\lambda_t)_t$ is a constant process (see Equation (8)). In the general case, as we only know that $(\lambda_t)_t$ is progressively measurable, the story is completely different. From Equation (7) and Proposition 1, if one has a bound on λ_t then SR^2 is also bounded. But the reverse is not automatically true. In fact, we will present in section 4.5 counter-examples which show that a price with constraint on SR^2 can be significatively greater than a price with constraint on $(\lambda_t)_t$, the risk premium of the non-hedgeable risk. Note that we will present in section 4.4 a definition of No Good Deal, local in time, due to Becherer (2009) and show that, in our context, it reduces to put a bound on $(\lambda_t)_t$ going back to Cochrane and Saa-Requejo (2001) and Björk and Slinko (2006) definitions.

We now define a good deal of level β for a pricing measure $\mathbb{Q} \in \mathcal{M}^e(\mathbb{P})$ named shortly as $GD(\beta, \mathbb{Q})$.

Definition 3 *Let $\beta > 0$ and $\mathbb{Q} \in \mathcal{M}^e(\mathbb{P})$, X is a $(\beta, \mathbb{Q})$-good deal $GD(\beta, \mathbb{Q})$ if $X \in \mathcal{C}(\mathbb{Q}) \cap \{X : \mathbb{E}(X) < \infty\}$ and $SR^2(X, \mathbb{Q}) > \beta$.*

Following the economic literature, β will be chosen around 2 but we will also provide in the numerical sections some studies on the variation of NGD price with respect to β. Following the No Good Deal literature, we will assume that :

Assumption 1 *There exists $\mathbb{Q} \in \mathcal{M}^2(\mathbb{P})$ and $\beta > 0$, such that there is no $(\beta, \mathbb{Q})$-good deal ($NGD(\beta, \mathbb{Q})$), i.e. for all $X \in \mathcal{C}(\mathbb{Q}) \cap \{X : \mathbb{E}(X) < \infty\}$, $SR^2(X, \mathbb{Q}) \le \beta$.*

Remark 1 *Under No Good Deal assumption there exists no claim such that $-X \in \mathcal{C}(\mathbb{Q}) \cap \{X : \mathbb{E}(X) < \infty\}$ and $SR^2(X, \mathbb{Q}) < -\beta$. Else $SR^2(-X, \mathbb{Q}) > \beta$ and $-X$ is a Good Deal.*

From Proposition 1, it follows that:

Theorem 1 *Assumption 1 is equivalent to the existence of some $\beta > 0$ such that*

$$\mathcal{M}^{2,\beta}(\mathbb{P}) := \left\{\mathbb{Q} \in \mathcal{M}^2(\mathbb{P}) \ : \ \|Z_T\|_{L^2(\mathbb{P})} \le \sqrt{1+\beta^2}\right\} \neq \emptyset.$$

Proof The first implication is a direct consequence of Proposition 1. Now assume that $\mathcal{M}^{2,\beta}(\mathbb{P}) \neq \emptyset$ for some $\beta > 0$ and choose $\mathbb{Q} \in \mathcal{M}^{2,\beta}(\mathbb{P})$. Then again by Proposition 1, for all $X_0 \in \mathcal{C}(\mathbb{Q}) \cap \{X : \mathbb{E}(X) < \infty\}$, $SR^2(X_0, \mathbb{Q}) \le \sup_{X \in \mathcal{C}(\mathbb{Q}) \cap \{X : \mathbb{E}(X) < \infty\}} SR^2(X, \mathbb{Q}) = \sqrt{\operatorname{Var} Z_T} \le \beta$, and $NGD(\beta, \mathbb{Q})$ holds.

4 Pricing Basis Risk

4.1 Introduction

In this section, we will investigate the notion of pricing for a contingent claim H depending on the non-traded asset V both from a theoretical and numerical point of view. Since the market is imperfect, this notion must be clarified and we will study and compute several notions of price.

A standard tool is the use of the super-replication price: intuitively, it is the minimal price which ensures in any situation the hedgeability of H. Mathematically, it is defined as the minimal initial wealth such that there exists a strategy leading to a terminal value almost surely over the claim. For example, for a call option, since H depends on the non-traded asset, one can show that the super-replication price is $+\infty$ if the investor is not endowed with at least one unit of V (else it is equal to V_0). [3]

Another classical price in the incomplete literature is the mean variance price that will be denoted by p_0^{MV}. This is the initial wealth that minimizes the quadratic hedging error (under the historical probability) at expiration, i.e. the difference between the claim we want to evaluate and the final value of his hedging strategy. This concept has been introduced by Föllmer and Sondermann (1986), in the martingale case. Further references and results on the optimal hedging strategy are given in section 5.3. From a pricing measure point of view, it can also be defined as the derivative's price computed under the minimal variance measure $\mathbb{Q}^0$ (see (5) with $\lambda = 0$)

$$p_0^{MV} = \mathbb{E}^{\mathbb{Q}^0}\left[\frac{(V_T - K)_+}{S_T^0}\right] = \mathbb{E}\left[e^{-rT} Z_T^0 (V_T - K)_+\right]. \tag{13}$$

In Theorem 3, we will compute it explicitly (see (23)), and in Theorem 5 we will show that it is actually the initial wealth that minimizes the quadratic hedging error.

We will compute in section 4.5 a quick and dirty approximations sometimes used in practice when dealing with Basis Risk and consisting in using the evolution property of S starting from V_0 to induce the evolution of V and price instead of the original option $(V_T - K)_+$, $\left(\frac{V_0}{S_0} S_T - K\right)_+$, whose underlying is now the tradable asset S.

And of course, we will study the prices induced by the No Good Deal criterium we have studied before. The one of Cochrane and Saa-Requejo (2001) and Björk and Slinko (2006), called CSR price (see (14)) and the one we propose called NGD price (see (17) and (18)). This is done in the next subsection. In particular, we will see that the NGD price is difficult to compute explicitly and we will provide upper and lower bounds p_0^{UB} and p_0^{LB} (see (22) and (20)) for it. We will present the criterium for No Good Deal of Becherer (2009) in section 4.4 and show that in our simple context, it reduces to CSR price.

All the numerical experiments are given in subsection 4.5 where we compute and compare the different prices mention above and in particular the CSR price and our upper and lower bounds p_0^{UB} and p_0^{LB} and exhibit various situations where our lower bound p_0^{LB}, and thus the NGD price, is significantly above the CSR price.

4.2 No Good Deal prices

The super-replication price has a so called dual representation ; it is equal to the supremum over all pricing measure $\mathbb{Q} \in \mathcal{M}^e(\mathbb{P})$ of the expectation of the discounted payoff, i.e. $\mathbb{E}^{\mathbb{Q}}(\frac{H}{S_T^0})$. The definition chosen by Cochrane and Saa-Requejo (2001) and Björk and Slinko (2006) following this dual setting is

$$p_0^{CSR}(H) = \sup_{\lambda \in L^2_{loc}((W,W^*)),\ s.t.\ \lambda \in [-\lambda^{max}, \lambda^{max}]} \mathbb{E}\left[Z_T^\lambda \frac{H}{S_T^0}\right], \tag{14}$$

where

$$\lambda^{max} = \sqrt{\frac{1}{T}\ln(1+\beta^2) - h_S^2}. \tag{15}$$

[3] In fact, if we start with a finite wealth X_0, since H depends on W^* through the non-traded V, we have that for any strategy (Φ^0, Φ), $\mathbb{P}[\Phi_T^0 S_T^0 + \Phi_T S_T < H] \neq 0$. Now if the investor is endowed with one unit of V : $\mathbb{P}(V_T \geq (V_T - K)_+) = 1$.

Note that from Equation (7), if $\lambda = (\lambda_t(\omega)) \in [-\lambda^{max}, \lambda^{max}]$ then $Z_T^\lambda \in \mathcal{M}^{2,\beta}(\mathbb{P})$ but the reverse is not true in general, as already mentioned in section 3.

No rigorous justification is given by the authors for the choice of pricing rule (14) as a dual representation of some No Good Deal price. To do so, we need to use the notion of coherent risk measure as already mention in Klöppel and Schweizer (2007) or Cherny (2008). We set u as the coherent utility function [4] related to the No Good Deal valuation, i.e.

$$u(X) = \inf_{\mathbb{Q}\in\mathcal{M}^{2,\beta}(\mathbb{P})} \mathbb{E}^{\mathbb{Q}}\left[\frac{X}{S_T^0}\right]. \tag{16}$$

The notion of hedgeability used in the super-replication price is now replaced by the notion of having a positive coherent utility : the No Good Deal upper-bound price is the minimal initial wealth such that there exists a strategy leading to a residual wealth having a positive utility. More precisely, denoting by $X_t^\Phi := \Phi_t^0 S_t^0 + \Phi_t^1 S_t$ for $\Phi \in \mathcal{A}_2$,

$$p_0(H) = \inf\left\{m \in \mathbb{R} \mid \exists\Phi \in \mathcal{A}_2 \text{ s.t. } X_0^\Phi = m \text{ and } u\left(X_T^\Phi - H\right) \geq 0\right\}. \tag{17}$$

Note that if u is identity we are back to the super-replication price definition.

Theorem 2 *Under the Assumption 1, the dual representation of the No Good Deal price defined in* (17) *is*

$$p_0(H) = \sup_{\mathbb{Q}\in\mathcal{M}^{2,\beta}(\mathbb{P})} \mathbb{E}^{\mathbb{Q}}\left[\frac{H}{S_T^0}\right]. \tag{18}$$

Proof Let $\Phi \in \mathcal{A}_2$, then using self financing (10), m being the initial value of the strategy Φ, we get that

$$X_T^\Phi = S_T^0\left(m + \int_0^T \Phi_t^1 d\frac{S_t}{S_t^0}\right).$$

Let $\mathbb{Q} \in \mathcal{M}^{2,\beta}(\mathbb{P})$, we get that $\mathbb{E}^{\mathbb{Q}}\left[\frac{X_T^\Phi}{S_T^0}\right] = m$.
And by (16), $u\left(X_T^\Phi - H\right) = m - \sup_{\mathbb{Q}\in\mathcal{M}^{2,\beta}(\mathbb{P})} \mathbb{E}^{\mathbb{Q}}\left[\frac{H}{S_T^0}\right]$. Thus,

$$p_0(H) = \inf\left\{m \in \mathbb{R} \mid \exists\Phi \in \mathcal{A}_2 \text{ s.t. } X_0^\Phi = m \text{ and } m \geq \sup_{\mathbb{Q}\in\mathcal{M}^{2,\beta}(\mathbb{P})} \mathbb{E}^{\mathbb{Q}}\left[\frac{H}{S_T^0}\right]\right\}.$$

Let $\Phi_t^0 = \sup_{\mathbb{Q}\in\mathcal{M}^{2,\beta}(\mathbb{P})} \mathbb{E}^{\mathbb{Q}}\left[\frac{H}{S_T^0}\right]$ and $\Phi_t^1 = 0$ for all t, then $\Phi \in \mathcal{A}_2$ and $X_0^\Phi = \sup_{\mathbb{Q}\in\mathcal{M}^{2,\beta}(\mathbb{P})} \mathbb{E}^{\mathbb{Q}}\left[\frac{H}{S_T^0}\right]$. Thus

$$p_0(H) = \sup_{\mathbb{Q}\in\mathcal{M}^{2,\beta}(\mathbb{P})} \mathbb{E}^{\mathbb{Q}}\left[\frac{H}{S_T^0}\right].$$

This concludes the proof.

Remark 2 *In the super-replication theory, there exists so called super-hedging strategies such that starting from the super-replication price and following some super-hedging strategy, H is fully hedged. But in the case of No Good Deal Pricing, no particular strategy appears : for example, Buy and Hold strategy in cash will do the job. We will discuss in section 5 some hedging criteria starting with an initial wealth equal to the No Good Deal price.*

[4] Since $\mathcal{M}^{2,\beta}(\mathbb{P})$ is non-empty, Theorem 2.2 of Cherny (2008) ensures that u is a so-called coherent utility function.

4.3 Explicit bounds for No Good deal price

The computation of the supremum in (18) is not easy. From Revuz and Yor (1994), the probability in the space $\mathcal{M}^{2,\beta}(\mathbb{P})$ can be represented by their densities, i.e. Z_T^λ (see (6)). But since λ is a stochastic process, this optimization problem is difficult to handle.

In Theorem 3, we propose to analyze the No Good Deal Price $p_0(H)$ (simply denote by p_0 from now). First, we provide some upper and lower bounds for p_0. The upper bound will be obtained by removing the positivity assumption and relaxing the martingale condition on the pricing measure density. To define our lower bound, we then assume that the risk premium λ of the non-hedgeable risk W^* is independent of the hedgeable risk W. This allows us to fully compute the optimization problem (18) when relaxing the positivity assumption on the pricing measure density. Then in order to obtain a equivalent martingale measure we just add the solution of (14).

Then we investigate the link between No Good Deal Price p_0 and the price p_0^{CSR} proposed by Cochrane and Saa-Requejo (2001) and by Björk and Slinko (2006) (see (14)). We show that p_0^{CSR} is always lower than our lower bound and thus the No Good Deal Price. We provide results for Call option but similar ones could be obtained for any claim H that has a closed form price in the Black and Scholes framework.

Theorem 3 *Let $H = (V_T - K)_+$. Assume that Assumption 1 holds and that $\frac{1}{T}\ln(1+\beta^2) \geq h_S^2$. There exists some lower bound p_0^{LB} and some upper bound p_0^{UB} such that*

$$p_0^{UB} \geq p_0 \geq p_0^{LB} \geq p_0^{CSR} > p_0^{MV}. \tag{19}$$

We have an explicit formulation for p_0^{LB}, p_0^{UB}, p_0^{MV} and p_0^{CSR} :

$$p_0^{LB} = \varepsilon p_0^{CSR} + (1-\varepsilon)e^{-rT}\mathbb{E}(Z_T^0 Y^{down} H), \tag{20}$$

where $\varepsilon \in (0,1)$, Y^{down} is defined in Lemma 2. Let

$$\bar{\beta} = \sqrt{(1+\beta^2)e^{-h_S^2 T} - 1}, \tag{21}$$

then

$$p_0^{UB} = p_0^{MV} + e^{-rT}e^{h_S^2 T/2}\bar{\beta}\sqrt{\mathbb{E}\left[H^2 - \mathbb{E}\left(H \mid \mathcal{F}_T^W\right)^2\right]}, \tag{22}$$

Moreover,

$$p_0^{MV} = e^{-rT}BS(V_0, T, K, \mu_V - \sigma_V\rho h_S, \sigma_V), \tag{23}$$
$$p_0^{CSR} = e^{-rT}BS(V_0, T, K, \mu_V - \sigma_V\rho h_S + \sigma_V\lambda^{max}\sqrt{1-\rho^2}, \sigma_V) \tag{24}$$

where the functional BS gives a kind of Black-Scholes price as a function of the initial price of the stock, the residual maturity and the strike of the option, the drift and the volatility of the stock : see Equation (56) in the Appendix for the precise definition and see (15) for the definition of λ^{max}.

Proof We first introduce a "degenerated" version of p_0, called $\hat{p}_0$, defined as the supremum of the discounted payoff over particular pricing measure in $\mathcal{M}^{2,\beta}(\mathbb{P})$. In fact, we assume that the risk premium process on non-coverable risk W^*, λ, is a constant number. Note that for this price, it is strictly equivalent to put a bound on λ or on SR^2, i.e. $\mathrm{Var}(Z_T)$ (see equation (8)).

$$\hat{p}_0(H) = \sup_{Z_T^\lambda \in \mathcal{M}^{2,\beta}(\mathbb{P})\ s.t.\ \lambda\in\mathbb{R}} \mathbb{E}\left[Z_T^\lambda \frac{H}{S_T^0}\right]$$

Step 1: Computation of $\hat{p}_0$
We begin by choosing a $\lambda \in \mathbb{R}$, and by computing

$$p_0^\lambda = \mathbb{E}\left[e^{-rT}Z_T^\lambda (V_T - K)_+\right]. \tag{25}$$

As λ is a constant process, we have seen (equation (8)) that $Z_T^\lambda \in \mathcal{M}^{2,\beta}(\mathbb{P})$ if and only if

$$\|Z_T\|_{L^2(\mathbb{P})} = e^{\frac{1}{2}(h_S^2+\lambda^2)T} \leq \sqrt{1+\beta^2} \Leftrightarrow \lambda \in [-\lambda^{max}, \lambda^{max}],$$

see (15) for the definition of λ^{max}. Thus,

$$\hat{p}_0 = \sup_{\lambda \in [-\lambda^{max}, \lambda^{max}]} p_0^\lambda.$$

From Girsanov Theorem (see for example Revuz and Yor (1994)), for any process $(\lambda_t)_t$,

$$W_t^\lambda := W_t + h_S t \text{ and } W_t^{\lambda,*} := W_t^* - \int_0^t \lambda_s ds \tag{26}$$

are standard Brownian motion under $\mathbb{Q}^\lambda$ defined by (5).
Thus, for all constant λ, the process V satisfy the stochastic differential equation:

$$dV_t = V_t((\mu_V - \sigma_V \rho h_S + \sigma_V \lambda \sqrt{1-\rho^2})dt + \sigma_V(\rho dW_t^\lambda + \sqrt{1-\rho^2} dW_t^{\lambda,*})). \tag{27}$$

We denote by η^λ the drift of this process, i.e.

$$\eta^\lambda = \mu_V - \sigma_V \rho h_S + \sigma_V \lambda \sqrt{1-\rho^2}.$$

From Appendix (56), we are able now to state that the quantity p_0^λ is given by a Black-Scholes type formula:

$$p_0^\lambda = e^{-rT} BS(V_0, T, K, \eta^\lambda, \sigma_V). \tag{28}$$

First, this allows us to compute p_0^{MV} :

$$p_0^{MV} = p_0^0 = \mathbb{E}\left[e^{-rT} Z_T^0 (V_T - K)_+\right] = e^{-rT} BS(V_0, T, K, \eta^0, \sigma_V).$$

Now we remark that BS is an increasing function of η (see Appendix (58)) and consequently p_0^λ is increasing in λ . Back to our optimization problem $\hat{p}_0$, we get that

$$\hat{p}_0 = \sup_{\lambda \in [-\lambda^{max}, \lambda^{max}]} p_0^\lambda = p_0^{\lambda^{max}} = e^{-rT} BS(V_0, T, K, \mu_V - \sigma_V \rho h_S + \sigma_V \lambda^{max} \sqrt{1-\rho^2}, \sigma_V),$$

and thus $\hat{p}_0 > p_0^{MV}$.

Step 2: Computation of p_0^{CSR}

The proof is based on a comparison theorem for the solution of stochastic differential equations. For a progressively predictable process λ_t, following the proof of step 1, we know that the process V^λ follows the SDE (27) replacing λ by λ_t. As $\lambda_t(\omega) \leq \lambda^{max}$, applying a comparison Theorem (see proposition 2.18 p.293 of Karatzas and Shreve (1991)), we obtain that $V_t^\lambda \leq \bar{V}_t$, $\mathbb{P}-p.s$, where the process $\bar{V}_t$ satisfies

$$d\bar{V}_t = \bar{V}_t \left((\mu_V - \sigma_V \rho h_S + \sigma_V \sqrt{1-\rho^2} \lambda^{max})dt + \sigma_V dU_t^*\right),$$

with $\bar{V}_0 = V_0$ and $U_t^* = \rho dW_t^\lambda + \sqrt{1-\rho^2} dW_t^{*,\lambda}$ a Brownian motion under the probability $\mathbb{Q}^\lambda$. Thus,

$$\mathbb{E}^{\mathbb{Q}^\lambda}\left[e^{-rT}(V_T^\lambda - K)_+\right] \leq \mathbb{E}^{\mathbb{Q}^\lambda}\left[e^{-rT}(\bar{V}_T - K)_+\right].$$

We compute the right hand side of this inequality using equation (56) (see Appendix) and obtain that

$$\mathbb{E}^{\mathbb{Q}^\lambda}\left[e^{-rT}(\bar{V}_T - K)_+\right] = e^{-rT} BS(V_0, T, K, \mu_V - \sigma_V \rho h_S + \sigma_V \lambda^{max} \sqrt{1-\rho^2}, \sigma_V)$$

Since $\hat{p}_0 \leq p_0^{CSR}$, step 1 shows that equality holds

$$p_0^{CSR} = \hat{p}_0 = e^{-rT} BS(V_0, T, K, \mu_V - \sigma_V \rho h_S + \sigma_V \lambda^{max} \sqrt{1-\rho^2}, \sigma_V). \tag{29}$$

Step 3: Definition and computation of p_0^{UB}

Define

$$p_0^{UB} = \sup_{\substack{Z,\ \mathbb{E}Z^2 \le 1+\beta^2 \\ \mathbb{E}\left(Z \mid \mathcal{F}_T^W\right) = Z_T^0}} \mathbb{E}\left[Z \frac{H}{S_T^0}\right] \tag{30}$$

Intuitively, this is an upper bound because we remove the positivity assumption and relax the martingale one on the pricing density Z. Note that the assumption $\mathbb{E}\left(Z \mid \mathcal{F}_T^W\right) = Z_T^0$ is equivalent to S/S^0 is martingale with respect to the hedgeable information only. We are going to prove that (22) holds.

p^{UB} is an upper bound: we show that any element $\mathbb{Q}^\lambda \in \mathcal{M}^{2,\beta}(\mathbb{P})$ satisfies the constraints of Problem 30. As $\mathbb{E}(Z_T^\lambda)^2 \le 1+\beta^2$, using (4), we get that

$$\mathbb{E}\left(Z_T^\lambda \mid \mathcal{F}_T^W\right) = Z_T^0 \mathbb{E}\left(Y_T^\lambda \mid \mathcal{F}_T^W\right) = Z_T^0 \mathbb{E}\left(1 + \int_0^T \lambda_t Y_t^\lambda dW_t^* \mid \mathcal{F}_T^W\right) = Z_T^0,$$

see, for the last equality, exercise 3.20 of Revuz and Yor (1994) p199.

Z^{UB} is the optimal solution of problem 30: Let

$$Z^{UB} = Z_T^0 + e^{h_S^2 T/2} \bar{\beta} \frac{H - \mathbb{E}\left(H \mid \mathcal{F}_T^W\right)}{\sqrt{\mathbb{E}\left[H^2 - \mathbb{E}\left(H \mid \mathcal{F}_T^W\right)^2\right]}}.$$

We show first that Z^{UB} satisfies constraints of Problem 30. It is straightforward that $\mathbb{E}\left(Z^{UB} \mid \mathcal{F}_T^W\right) = Z_T^0$. Furthermore, since $H - \mathbb{E}\left(H \mid \mathcal{F}_T^W\right)$ is orthogonal to $\mathcal{F}_T^W$ and thus orthogonal to Z_T^0 :

$$\begin{aligned}
\mathbb{E}(Z^{UB})^2 &= \mathbb{E}(Z_T^0)^2 + e^{h_S^2 T} \bar{\beta}^2 \frac{\mathbb{E}\left[H - \mathbb{E}\left(H \mid \mathcal{F}_T^W\right)\right]^2}{\mathbb{E}\left[H^2 - \mathbb{E}\left(H \mid \mathcal{F}_T^W\right)^2\right]} = \mathbb{E}(Z_T^0)^2 + e^{h_S^2 T} \bar{\beta}^2 \\
&= e^{h_S^2 T} + (1+\beta^2) - e^{h_S^2 T} = 1 + \beta^2,
\end{aligned}$$

because $\mathbb{E}\left[H\mathbb{E}(H \mid \mathcal{F}_T^W)\right] = \mathbb{E}\left[\mathbb{E}\left(H \mid \mathcal{F}_T^W\right)\right]^2$ and using successively (8) and (21).
Now, we prove that Z^{UB} reaches the maximal value of Problem 30.

$$\begin{aligned}
\mathbb{E}\left(Z^{UB} H\right) &= \mathbb{E}\left(Z_T^0 H\right) + e^{h_S^2 T/2} \bar{\beta} \frac{\mathbb{E}\left(H^2\right) - \mathbb{E}\left[H\mathbb{E}(H \mid \mathcal{F}_T^W)\right]}{\sqrt{\mathbb{E}\left[H^2 - \mathbb{E}\left(H \mid \mathcal{F}_T^W\right)^2\right]}} \\
&= \mathbb{E}\left(Z_T^0 H\right) + e^{h_S^2 T/2} \bar{\beta} \sqrt{\mathbb{E}\left[H^2 - \mathbb{E}\left(H \mid \mathcal{F}_T^W\right)^2\right]}.
\end{aligned}$$

Let Z such that $\mathbb{E}Z^2 \le 1+\beta^2$ and $\mathbb{E}\left(Z \mid \mathcal{F}_T^W\right) = Z_T^0$, we get that

$$\begin{aligned}
\mathbb{E}(ZH) &= \mathbb{E}\left[Z\left(H - \mathbb{E}\left(H \mid \mathcal{F}_T^W\right)\right)\right] + \mathbb{E}\left[Z\mathbb{E}\left(H \mid \mathcal{F}_T^W\right)\right] \\
&= \mathbb{E}\left[(Z - Z_T^0)\left(H - \mathbb{E}\left(H \mid \mathcal{F}_T^W\right)\right)\right] + \mathbb{E}\left[\mathbb{E}\left(Z \mid \mathcal{F}_T^W\right)\mathbb{E}\left(H \mid \mathcal{F}_T^W\right)\right] \\
&\le \sqrt{\mathbb{E}\left[Z - Z_T^0\right]^2} \sqrt{\mathbb{E}\left[H - \mathbb{E}\left(H \mid \mathcal{F}_T^W\right)\right]^2} + \mathbb{E}\left(Z_T^0 H\right) \\
&\le \mathbb{E}\left(Z_T^0 H\right) + e^{h_S^2 T/2} \bar{\beta} \sqrt{\mathbb{E}\left[H - \mathbb{E}\left(H \mid \mathcal{F}_T^W\right)\right]^2} = \mathbb{E}\left(Z^{UB} H\right),
\end{aligned}$$

where we have use successively that Z_T^0 is orthogonal to $H - \mathbb{E}\left(H \mid \mathcal{F}_T^W\right)$, Cauchy-Schwartz inequality and (8)

$$\begin{aligned}
\mathbb{E}\left[Z - Z_T^0\right]^2 &= \mathbb{E}(Z)^2 - 2\mathbb{E}\left(Z Z_T^0\right) + \mathbb{E}\left(Z_T^0\right)^2 \\
&\le 1 + \beta^2 - 2\mathbb{E}\left(\mathbb{E}\left(Z \mid \mathcal{F}_T^W\right) Z_T^0\right) + \mathbb{E}\left(Z_T^0\right)^2 \\
&\le 1 + \beta^2 - \mathbb{E}\left(Z_T^0\right)^2 = 1 + \beta^2 - e^{h_S^2 T} = e^{h_S^2 T} \bar{\beta}^2
\end{aligned}$$

Step 4: Definition and computation of p_0^{LB}

The definition of the lower bound is a little more tricky. We first reformulate our problem using the probability $\mathbb{Q}^{Z^0}$. $\mathbb{Q}^{Z^0}$ is defined by $d\mathbb{Q}^{Z^0}/d\mathbb{P} = (Z_T^0)^2/\mathbb{E}(Z_T^0)^2 = (Z_T^0)^2 e^{-h_S^2 T}$ (see (8)). This is the so called minimum variance probability (see section 5.3). It will be used in quadratic hedging part of the paper. Using Bayes formula and recalling definition of Y_T^λ (see (4)), we get that

$$\mathbb{E}\left[Z_T^\lambda \frac{H}{S_T^0}\right] = e^{-rT}\mathbb{E}\left[(Z_T^0)^2 Y_T^\lambda \frac{H}{Z_T^0}\right] = e^{(h_S^2-r)T}\mathbb{E}^{\mathbb{Q}^{Z^0}}\left[Y_T^\lambda \frac{H}{Z_T^0}\right]$$

We now rewrite the constraints of problem (18) :

$$\mathbb{E}\left[(Z_T)^2\right] = \mathbb{E}\left[(Z_T^0)^2 (Y_T^\lambda)^2\right] = e^{h_S^2 T}\mathbb{E}^{\mathbb{Q}^{Z^0}}\left[(Y_T^\lambda)^2\right]$$

Using the definition of $\bar{\beta}$ (21), we get that

$$\mathbb{E}\left[(Z_T^\lambda)^2\right] \leq 1+\beta^2 \Leftrightarrow \mathbb{E}^{\mathbb{Q}^{Z^0}}\left[(Y_T^\lambda)^2\right] \leq 1+\bar{\beta}^2. \tag{31}$$

We now assume that λ, the risk premium of the non-hedgeable risk W^*, is independent of the hedgeable risk W. Then $Y_T^\lambda \in \mathcal{F}_T^{W^*}$ and thus

$$\mathbb{E}\left[Z_T^\lambda \frac{H}{S_T^0}\right] = e^{(h_S^2-r)T}\mathbb{E}^{\mathbb{Q}^{Z^0}}\left[Y_T^\lambda \frac{H}{Z_T^0}\right] = e^{(h_S^2-r)T}\mathbb{E}^{\mathbb{Q}^{Z^0}}\left[Y_T^\lambda \mathbb{E}^{\mathbb{Q}^{Z^0}}\left[\frac{H}{Z_T^0} \mid \mathcal{F}_T^{W^*}\right]\right].$$

We now compute $\mathbb{E}^{\mathbb{Q}^{Z^0}}\left[\frac{H}{Z_T^0} \mid \mathcal{F}_T^{W^*}\right]$ using $\mathbb{Q}^0$. As $d\mathbb{Q}^{Z^0}/d\mathbb{Q}^0 = Z_T^0 e^{-h_S^2 T}$,

$$\mathbb{E}^{\mathbb{Q}^{Z^0}}\left[\frac{H}{Z_T^0} \mid \mathcal{F}_T^{W^*}\right] = \frac{\mathbb{E}^{\mathbb{Q}^0}\left[e^{-h_S^2 T} Z_T^0 \frac{H}{Z_T^0} \mid \mathcal{F}_T^{W^*}\right]}{\mathbb{E}^{\mathbb{Q}^0}\left[e^{-h_S^2 T} Z_T^0 \mid \mathcal{F}_T^{W^*}\right]} = e^{-h_S^2 T}\mathbb{E}^{\mathbb{Q}^0}\left[H \mid \mathcal{F}_T^{W^*}\right],$$

because $\mathbb{E}^{\mathbb{Q}^0}\left[e^{-h_S^2 T} Z_T^0 \mid \mathcal{F}_T^{W^*}\right] = e^{-h_S^2 T}\mathbb{E}^{\mathbb{Q}^0}\left[Z_T^0\right] = e^{-h_S^2 T}\mathbb{E}\left[\left(Z_T^0\right)^2\right] = 1$.
$\mathbb{E}^{\mathbb{Q}^0}\left[H \mid \mathcal{F}_T^{W^*}\right]$ is fully calculable. If we rewrite V_t with $\mathbb{Q}^0$ (see (27)):

$$V_t = V_0 e^{\sigma_V\sqrt{1-\rho^2}W_T^*} \times e^{\left(\mu_V - \sigma_V\rho h_S - \frac{\sigma_V^2}{2}\right)T + \sigma_V\rho W_T^0}.$$

Then,

$$\mathbb{E}^{\mathbb{Q}^0}\left[H \mid \mathcal{F}_T^{W^*}\right] = \mathbb{E}^{\mathbb{Q}^0}\left[(V_T-K)_+ \mid \mathcal{F}_T^{W^*}\right] = \psi(W_T^*), \tag{32}$$

with

$$\begin{aligned}\psi(x) &= \mathbb{E}^{\mathbb{Q}^0}\left[\left(V_0 e^{\sigma_V\sqrt{1-\rho^2}x} \times e^{\left(\mu_V-\sigma_V\rho h_S-\frac{\sigma_V^2}{2}\right)T+\sigma_V\rho W_T^0} - K\right)_+\right] \\ &= \mathbb{E}^{\mathbb{Q}^0}\left[\left(V_0 e^{\sigma_V\sqrt{1-\rho^2}x} e^{\left(\mu_V-\sigma_V\rho h_S-(1-\rho^2)\frac{\sigma_V^2}{2}-\rho^2\frac{\sigma_V^2}{2}\right)T+\sigma_V\rho W_T^0} - K\right)_+\right].\end{aligned}$$

Therefore ψ can be expressed with a Black-Scholes type formula (see (56) in Appendix):

$$\psi(x) = BS\left(V_0 e^{\sigma_V\sqrt{1-\rho^2}x}, T, K, \mu_V - \sigma_V\rho h_S - (1-\rho^2)\frac{\sigma_V^2}{2}, \sigma_V\rho\right).$$

Note that we get similarly, for all $Y \in \mathcal{F}_T^{W^*}$

$$\mathbb{E}\left[Z_T^0 Y \frac{H}{S_T^0}\right] = e^{-rT}\mathbb{E}^{\mathbb{Q}^{Z^0}}\left[Y\psi(W_T^*)\right] \tag{33}$$

So going back to our optimization problem

$$\begin{aligned}
p_0 \;\geq & \sup_{\substack{\mathbb{Q}^\lambda \in \mathcal{M}^{2,\beta}(\mathbb{P}) \\ \lambda_t \in \mathcal{F}_t^{W^*}}} e^{-rT}\mathbb{E}^{\mathbb{Q}^\lambda}\left[(V_T-K)_+\right] \\
= & \sup_{\substack{\lambda_t \in \mathcal{F}_t^{W^*} \\ \mathbb{E}^{\mathbb{Q}^{Z^0}}\left(Y_T^\lambda\right)^2 \leq 1+\bar{\beta}^2}} e^{-rT}\mathbb{E}^{\mathbb{Q}^{Z^0}}\left[Y_T^\lambda \psi(W_T^*)\right] \qquad (34)\\
=^{5} & \sup_{\substack{Y>0,\ \mathbb{E}^{\mathbb{Q}^{Z^0}} Y=1 \\ \mathbb{E}^{\mathbb{Q}^{Z^0}} Y^2 \leq 1+\bar{\beta}^2}} e^{-rT}\mathbb{E}^{\mathbb{Q}^{Z^0}}\left[Y\psi(W_T^*)\right] \qquad (35)
\end{aligned}$$

We now able to state our new optimization problem used for the computation of the lower bound of Problem 18 (we relax the positivity Assumption on Y).

$$p_0^{down} = \sup_{\substack{Y\geq 0,\ \mathbb{E}^{\mathbb{Q}^{Z^0}} Y=1 \\ \mathbb{E}^{\mathbb{Q}^{Z^0}} Y^2 \leq 1+\bar{\beta}^2}} e^{-rT}\mathbb{E}^{\mathbb{Q}^{Z^0}}\left[Y\psi(W_T^*)\right] \qquad (36)$$

Let Y^{down} be the solution of problem 36 (see Lemma 2 below) then $\varepsilon Y_T^{\lambda^{max}}+(1-\varepsilon)Y^{down}$ satisfies conditions of problem 35 (the two first conditions are obviously satisfied and the third one comes directly from Cauchy-Schwartz inequality) for some $\varepsilon>0$. From (33),

$$e^{-rT}\mathbb{E}^{\mathbb{Q}^{Z^0}}\left[Y^{down}\psi(W_T^*)\right] = e^{-rT}\mathbb{E}\left[Z_T^0 Y^{down}H\right]$$

and recalling (33), (25), (28) and (29), we get that

$$\begin{aligned}
e^{-rT}\mathbb{E}^{\mathbb{Q}^{Z^0}}\left[Y_T^{\lambda^{max}}\psi(W_T^*)\right] &= e^{-rT}\mathbb{E}[Z_T^0 Y_T^{\lambda^{max}}H] = p_0^{\lambda^{max}} \\
&= e^{-rT}BS(V_0,T,K,\eta^{\lambda^{max}},\sigma_V) = p_0^{CSR}
\end{aligned}$$

So we have found a lower bound for p_0 :

$$p_0 \geq \varepsilon p_0^{CSR} + (1-\varepsilon)e^{-rT}\mathbb{E}[Z_T^0 Y^{down}H]$$

It remains to find a solution for problem 36 : this is done in the lemma below which proof is postponed in Appendix 6.1.3.

Lemma 2 *The solution of Problem 36 is :*
if $1-\bar{\beta}\frac{\mathbb{E}^{\mathbb{Q}^{Z^0}}(\psi(W_T^*))}{\sqrt{\mathrm{Var}^{\mathbb{Q}^{Z^0}}\psi(W_T^*)}} \geq 0$ *then*

$$Y^{down} = 1+\bar{\beta}\frac{\psi(W_T^*)-\mathbb{E}^{\mathbb{Q}^{Z^0}}(\psi(W_T^*))}{\sqrt{\mathrm{Var}^{\mathbb{Q}^{Z^0}}\psi(W_T^*)}} \text{ and } p_0^{down} = \mathbb{E}\left(\psi(W_T^*)\right)+\bar{\beta}\sqrt{\mathrm{Var}^{\mathbb{Q}^{Z^0}}\psi(W_T^*)}.$$

[5] The equality between problems 34 and 35 comes from the following observations : let Y^{opt} be the solution of problem 35, and $Y_t^{opt}=\mathbb{E}^{\mathbb{Q}^{Z^0}}\left(Y^{opt}\mid\mathcal{F}_t^{W^*}\right)$. As $(Y_t^{opt})_t$ is a L^2 $(\mathbb{Q}^{Z^0},\mathcal{F}^{W^*})$-martingale, from Theorem of martingale representation (see for example Revuz and Yor (1994)) there exists $k_t\in L^2_{loc}(W^*)$ such that $dY_t^{opt}=k_t dW_t^*$. Let $\lambda_t^{opt}=k_t/Y_t^{opt}$ (note that $Y_t^{opt}>0$), $\lambda_t^{opt}\in\mathcal{F}_t^{W^*}$. By Ito formula $Y_T^{opt}=Y_0^{opt}+\int_0^T\lambda_t^{opt}Y_t^{opt}dW_t^*=1+\int_0^T\lambda_t^{opt}Y_t^{opt}dW_t^*=Y_T^{\lambda^{opt}}$. Thus λ^{opt} satisfies condition of 34 and problem 35 is lower than problem 34. Let λ satisfying condition of 34 then Y_T^λ satisfies condition of 35 and thus the two problems are equal.

if $1-\bar{\beta}\frac{\mathbb{E}^{\mathbb{Q}^{Z^0}}(\psi(W_T^*))}{\sqrt{\mathrm{Var}^{\mathbb{Q}^{Z^0}}\psi(W_T^*)}}<0$ *then*

$$Y^{down}=\frac{(\psi(W_T^*)-\alpha)_+}{\mathbb{E}^{\mathbb{Q}^{Z^0}}(\psi(W_T^*)-\alpha)_+}\ \textit{and}\ p_0^{down}=\alpha+(1+\bar{\beta})^2\mathbb{E}^{\mathbb{Q}^{Z^0}}(\psi(W_T^*)-\alpha)_+,$$

where there exists a positive number α *such that* $\frac{\mathbb{E}^{\mathbb{Q}^{Z^0}}(\psi(W_T^*)-\alpha)_+^2}{\left(\mathbb{E}^{\mathbb{Q}^{Z^0}}\left(\psi(W_T^*)-\alpha\right)_+\right)^2}=1+\bar{\beta}^2$.

Step 5: Proof of $p_0^{LB}\geq p_0^{CSR}$ First as already note $Y_T^{\lambda^{max}}$ satisfies the constraints of Problem 36 which implies that $p_0^{down}\geq p_0^{CSR}$. Thus, using the definition of p_0^{LB}

$$p_0^{LB}-p_0^{CSR}=(1-\varepsilon)(p_0^{down}-p_0^{CSR})\geq 0.$$

4.4 Dynamic No Good Deal pricing

In the literature, in particular in Klöppel and Schweizer (2007) and Becherer (2009), dynamics conditions of No Good Deals are proposed. We will focus on the work of Becherer (2009) since he proposes a hedging criterium linked to his No Good Deal price. In order to compare his condition with our No Good Deal principle, we first present its dynamic formulation. For a measure $\mathbb{Q}\in\mathcal{M}^2(\mathbb{P})$, let

$$SR_t^2(X,\mathbb{Q})=\frac{\mathbb{E}(X|\mathcal{F}_t)-\mathbb{E}^{\mathbb{Q}}(X|\mathcal{F}_t)}{\sqrt{\mathrm{Var}(X|\mathcal{F}_t)}}. \tag{37}$$

With $\mathcal{C}_t(\mathbb{Q})=\{X\in L^0(\mathbb{P})\ :X^-\in L^\infty(\mathbb{P})\text{ and }\mathbb{E}^{\mathbb{Q}}(X|\mathcal{F}_t)<\infty\}$, similarly as in Proposition 1, for $\mathbb{Q}\in\mathcal{M}^2(\mathbb{P})$ we get that

$$\operatorname{ess}\sup_{X\in\mathcal{C}_t(\mathbb{Q})\cap\{X\ :\mathbb{E}(X|\mathcal{F}_t)<\infty\}}SR_t^2(X,\mathbb{Q})=\sqrt{\mathrm{Var}\left(\frac{Z_T}{Z_t}|\mathcal{F}_t\right)}\quad a.s.,$$

and we can define the dynamic No Good deal set by

$$\mathcal{M}^{2,\beta,dyn}(\mathbb{P}):=\left\{\mathbb{Q}\in\mathcal{M}^2(\mathbb{P})\ :\ \mathbb{E}\left(\left(\frac{Z_T}{Z_t}\right)^2|\mathcal{F}_t\right)\leq 1+\beta_t^2\quad a.s.,\ \forall t<T\right\}\text{ with} \tag{38}$$

$$\beta_t=\sqrt{(1+\beta^2)^{1-\frac{t}{T}}-1}. \tag{39}$$

We choose this definition for β_t because it is coherent with definition 3 : for $t=0$, we go back to $\mathbb{E}\left((Z_T)^2\right)\leq 1+\beta^2$ which is Theorem 1.

We will define our No Good deal price as before : let u_t be the coherent utility function related to the No Good Deal valuation (see also Klöppel and Schweizer (2007)),

$$u_t(X)=\operatorname{ess}\inf_{\mathbb{Q}\in\mathcal{M}^{2,\beta,dyn}(\mathbb{P})}\mathbb{E}^{\mathbb{Q}}\left[\frac{X}{S_T^0}|\mathcal{F}_t\right]. \tag{40}$$

Then

$$p_t(H)=\operatorname{ess}\inf\left\{X_t\ \middle|\ \exists\Phi\in\mathcal{A}_2\text{ s.t. }X_t^\Phi=X_t\text{ and }u_t\left(X_T^\Phi-H\right)\geq 0\quad a.s.\right\}. \tag{41}$$

Then under the Assumption that $\mathcal{M}^{2,\beta,dyn}(\mathbb{P}) \neq \emptyset$, the dual representation of the dynamic No Good Deal price defined in (41) is

$$p_t(H) = \text{ess} \sup_{\mathbb{Q}\in\mathcal{M}^{2,\beta,dyn}(\mathbb{P})} S_t^0 \mathbb{E}^{\mathbb{Q}} \left[\frac{H}{S_T^0} | \mathcal{F}_t \right] \quad a.s. \,. \tag{42}$$

The proof is very similar of the one of (18) and thus omitted. The generalization of the definition chosen by Cochrane and Saa-Requejo (2001) and Björk and Slinko (2006) is the following

$$p_t^{CSR}(H) = \text{ess} \sup_{\mathbb{Q}^\lambda\in\mathcal{M}^{2,\beta,dyn}(\mathbb{P}),\ s.t.\ (\lambda_t)\in[-\lambda^{max},\lambda^{max}]} S_t^0 \mathbb{E} \left[Z_T^\lambda \frac{H}{S_T^0} | \mathcal{F}_t \right] \quad a.s. \,, \tag{43}$$

see (15) for the definition of λ^{max}.

Theorem 4 *Let $H = (V_T - K)_+$. Assume that $\mathcal{M}^{2,\beta,dyn}(\mathbb{P}) \neq \emptyset$ and that $\frac{1}{T}\ln(1+\beta^2) \geq h_S^2$. Then*

$$p_t(H) \geq p_t^{CSR} = e^{-r(T-t)} BS(V_t, T-t, K, \mu_V - \sigma_V \rho h_S + \sigma_V \lambda^{max} \sqrt{1-\rho^2}, \sigma_V) \quad a.s. \tag{44}$$

where the functional BS give a kind of Black-Scholes price as a function of the initial price of the stock, the residual maturity and the strike of the option, the drift and the volatility of the stock : see Equation (56) *in the Appendix for the precise definition and see (15) for the definition of λ^{max}.*

Proof Using the same line of arguments as in step 1 and 2 of Theorem 3, it is easy to see that $p_t^{CSR}(H)$ is also equal to the supremum of the discounted payoff over pricing measure in $\mathcal{M}^{2,\beta,dyn}(\mathbb{P})$ such that λ is a deterministic process. In fact, in this case

$$S_t^0 \mathbb{E}^{Q^\lambda} \left[\frac{H}{S_T^0} | \mathcal{F}_t \right] = e^{-r(T-t)} BS \left(V_t, T-t, K, \mu_V - \sigma_V \rho h_S + \sigma_V \sqrt{1-\rho^2} \frac{1}{T-t} \int_t^T \lambda_s ds, \sigma_V \right).$$

As BS is increasing in his drift term, the maximum of $S_t^0 \mathbb{E}^{Q^\lambda} \left[\frac{H}{S_T^0} | \mathcal{F}_t \right]$ on λ will be attained for maximum value of $\frac{1}{T-t}\int_t^T \lambda_s ds$. Remarking that $\mathbb{E}\left(\left(\frac{Z_T^\lambda}{Z_t^\lambda} \right)^2 | \mathcal{F}_t \right) = e^{h_S^2(T-t) + \int_t^T \lambda_s^2 ds}$, the Dynamic No Good Deal condition implies that $\frac{1}{T-t}\int_t^T \lambda_s^2 ds \leq (\lambda^{max})^2$ (see (15) for the definition of λ^{max}). As from Cauchy Schwartz inequality, $\frac{1}{T-t}\int_t^T \lambda_s ds \leq \sqrt{\frac{1}{T-t}\int_t^T \lambda_s^2 ds}$, the maximum in

$$\text{ess} \sup_{\mathbb{Q}^\lambda\in\mathcal{M}^{2,\beta,dyn}(\mathbb{P}),\ s.t.\ \lambda_t\in\mathbb{R}} S_t^0 \mathbb{E} \left[Z_T^\lambda \frac{H}{S_T^0} \right]$$

is also attains by λ^{max}. Then the second step follows and the proof is complete.

We now turn to Becherer (2009) definition : he generalizes the No Good Deal principle be mean of conditions, local in time, on the relative entropy of the martingale measure : namely, for a given process l and all stopping time τ_1, τ_2 with $\tau_1 \leq \tau_2 \leq T$:

$$\mathbb{E}\left(-\log \frac{Z_{\tau_2}}{Z_{\tau_1}} | \mathcal{F}_{\tau_1} \right) \leq \frac{1}{2} \mathbb{E}\left(\int_{\tau_1}^{\tau_2} l_s ds | \mathcal{F}_{\tau_1} \right). \tag{45}$$

In order to be coherent with our setup, we choose l constant equal to $\sqrt{h_S^2 + (\lambda_{max})^2} = \sqrt{\frac{1}{T}\ln(1+\beta^2)}$, then from Proposition 4.0.11 in Becherer (2009), we get that a martingale measure $\mathbb{Q}^\lambda$ satisfies the No Good Deal principle defined in (45) if and only if $|\lambda_t| \leq \lambda^{max}$. Moreover, it is easy to see (see his Theorem 5.0.15) that in this context, the NGD price of Becherer (2009) is given by (44). So we are back to the definition given by Cochrane and Saa-Requejo (2001). This shows that, in our setup, the condition given by Becherer (2009) generalizes the CSR condition and turns again to bound the risk premium of the non hedgeable risk.

4.5 NGD price can be strictly higher than CSR price

4.5.1 Framework

We will compute and compare the CSR price defined by Cochrane and Saa-Requejo (2001), (see (14)) and our upper and lower bounds p_0^{UB} and p_0^{LB} (see (22) and (20)) and exhibit various situations where our lower bound p_0^{LB}, and thus the NGD price (see (18)), is significantly above the CSR price. In all figures, our lower bound will be denoted by "NGD-LB", ou upper bound by "NGD-UB", the CSR price by "NGD-CSR". By extension, we will call them the NGD prices. The two main parameters of our problem are β and the correlation ρ. The first one measures if a strategy is a good deal and thus has to be excluded from the market and the second one measures the similarity of the two assets V and S. We will show numerically and analytically the convergence of the NGD prices with respect to both parameters.

We will also compare our results with different notions of prices. The minimum variance price (see (13) and (23)) that we have already characterized (in all figures it will be denoted by "MV-Price"). We will also consider a price which is used sometimes in practice when dealing with Basis Risk. As we consider that the processes V and S are highly correlated, as a quick and dirty approximation, one can use the evolution property of S (i.e. the drift μ_S and the volatility σ_S) starting from V_0 to induce the evolution of V and price instead of our original option $(V_T - K)_+$, $\left(\frac{V_0}{S_0} S_T - K\right)_+$, whose underlying is now the tradable asset S. Therefore, one can approximate the option price by the Black-Scholes price of this claim, denoted $S\text{-}BS_t$:

$$S\text{-}BS_t = e^{-r(T-t)} \frac{V_0}{S_0} BS\left(S_t, T-t, K\frac{S_0}{V_0}, r, \sigma_S\right). \tag{46}$$

In the sequel, it will be designated as the "S-BS Price".
Finally, we look at the "real" Black-Scholes price of the contingent claim, denoted by $V\text{-}BS_t$. This is the price of an option on V in a market constituted by S^0 and V, when V is tradable. Of course, this price has no economic sense in case of Basis Risk. It is given by

$$V\text{-}BS_t = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}^V}\left[(V_T - K)_+ \mid \mathcal{F}_t\right] = e^{-r(T-t)} BS(V_t, T-t, K, r, \sigma_V),$$

where the probability $\mathbb{Q}^V$ is the martingale probability for V, i.e.

$$\frac{d\mathbb{Q}^V}{d\mathbb{P}} = \exp\left(-h_V(\rho W_T + \sqrt{1-\rho^2} W_T^*) - \frac{1}{2} h_V^2 T\right).$$

As the preceding, we will note it "V-BS Price".
We will perform our computation for the set of parameters described in table 1. The parameters

μ_V	σ_V	V_0	μ_S	σ_S	S_0	r	T
0.04	0.32	15	0.0272	0.256	100	2%	0.25

Table 1: *Set of parameters*

are choose such that V is more risky than S (i.e. the volatility is higher) but provides a higher return (the drift is also higher). We choose to start from a different initial stock value as this is the case for a stock and an index, for example. But experiments performed with similar initial stock value lead to the similar conclusions.

Economic literature asserts that a reasonable value for β is 2 (see for example Cochrane and Saa-Requejo (2001)) and as we are interested in hedging basis risk, we will choose assets which are well correlated, $\rho = 0.8$ at least.

In figure 1, we plot the different prices w.r.t ρ for three different values of β : 0.6, 2 and 3.4, and different values of K (at, in and out the money, i.e. $K = 15, 10$ or 20). The correlation ρ belongs to $[0.1, 0.95]$ with step size of 0.05. Figure 2 shows the price as a function of β with fixed ρ. We choose β in $[0.6; 3.4]$ with a step size of 0.2 and ρ equals successively 0.2, 0.5 and 0.8.

4.5.2 Prices comparaison

We observe first that the NGD prices (p_0^{UB}, p_0^{LB} and p_0^{CSR}) are considerably smaller than the initial value of the stock, which is equal to 15 in our example. Note that, this bound is not always the super-replication price because the underlying asset is not tradable. But in the case where the investor is endowed with a unit of V, it is clearly the super-replication cost.

Next, it appears clearly in figure 1 and 2 that the CSR price p_0^{CSR} is strictly below our lower bound p_0^{LB} and thus the NGD price. For the following situation, which is economically meaningful : highly correlated assets ($\rho = 0.8$) and $\beta = 2$ and an at the money option, $p_0^{UB} = 3.1$ and $p_0^{LB} = 2.59$ while $p_0^{CSR} = 2.37$: thus the lower bound is 8.4% over the CSR price. Beside this economically classical case, we note that the CSR price is strictly smaller than lower bound p_0^{LB} in most of the cases especially when the option is at and out the money. The gap between both prices is 21% for $\beta = 3.4$, $K = 17.5$ and $\rho = 0.4$ ($p_0^{CSR} = 2.136$ while $p_0^{LB} = 2.597$) and can even reach a value of 25%.

With our set of parameters, the “V-BS Price” is close to the “MV-Price” but this is not true in general. For example if we put $\sigma_S = 0.02$ and $\rho = 0.8$, “MV-Price”= 0.08 while “V-BS Price”= 0.48. Similarly, “S-BS Price” is very low in our example. This comes from the choice of the volatility of V which is much higher than those of S (recall that BS function is increasing with volatility). Thus, in our example, “S-BS Price” clearly underestimate the price of the option.

4.5.3 Variation of prices with ρ and β

Figures 1 and 2 show that NGD prices (p_0^{UB}, p_0^{LB} and p_0^{CSR}) decrease with ρ and increase with β. This is true theoretically for CSR price. Recall formula (24) and observe that “BS” is an increasing function of the drift (see appendix (58)) and that this drift, equal to $\mu_V - \sigma_V \rho h_S + \sigma_V \lambda^{max}\sqrt{1-\rho^2}$ (see (15) for the definition of λ^{max}), is clearly increasing in β and decreasing in ρ. It is also clear that p_0^{UB} increases with β : see equation (22). The growth of p_0^{UB} in ρ or the variation of p_0^{LB} in both parameters are theoretically less clear.

Let us consider the limit cases in β. Note that the NGD price p_0 is clearly increasing in β as $\mathcal{M}^{2,\beta}(\mathbb{P})$ (see Theorem 1 and (18)) : when β goes to infinity, $\mathcal{M}^{2,\beta}(\mathbb{P})$ tends to $\mathcal{M}^2(\mathbb{P})$ and thus p_0 converges to the super-replication price. In the opposite case (in Theorem 3), we assume that $\beta \geq \sqrt{e^{h_S^2 T} - 1} = 0.014$ with the chosen parameters. In this limit case, $\bar{\beta} = 0$ (see (21)) and $\lambda^{max} = 0$ (see (15)) thus $p_0^{CSR} = p_0^{\lambda^{max}} = p_0^0 = p_0^{UB}$ (see (22), (23) and (24)). It follows that all No Good Deal prices : p_0^{UB}, p_0, p_0^{CSR} and p_0^{LB} (from (19)) converge to MV Price as observed in figure 2 when β goes to $\sqrt{e^{h_S^2 T} - 1} = 0.014$.

We also observe that all No Good Deal prices (including p_0) converge to MV price when $\rho \to 1$. We show below that this is theoretically correct. It is quite clear from (23) and (24) that the CSR price converges to MV price. But without referring to our Theorem, when $\rho = 1$, the non-exchangeable asset V depends only on W, which is now the single source of risk : the market is complete and the set $\mathcal{M}^2(\mathbb{P})$ is reduced to $\mathbb{Q}^0$. Thus, the contingent claim $(V_T - K)_+$ is perfectly replicable and it price under the unique equivalent measure $\mathbb{Q}^0$ is the MV Price. For the upper bound, since H depends only on W, $\mathbb{E}[H|\mathcal{F}_T^W] = H$ and thus $p_0^{UB} = p_0^{MV}$ (see (22)). From (19), we deduce the convergence result for the lower bound p_0^{LB} and also for the NGD price p_0.

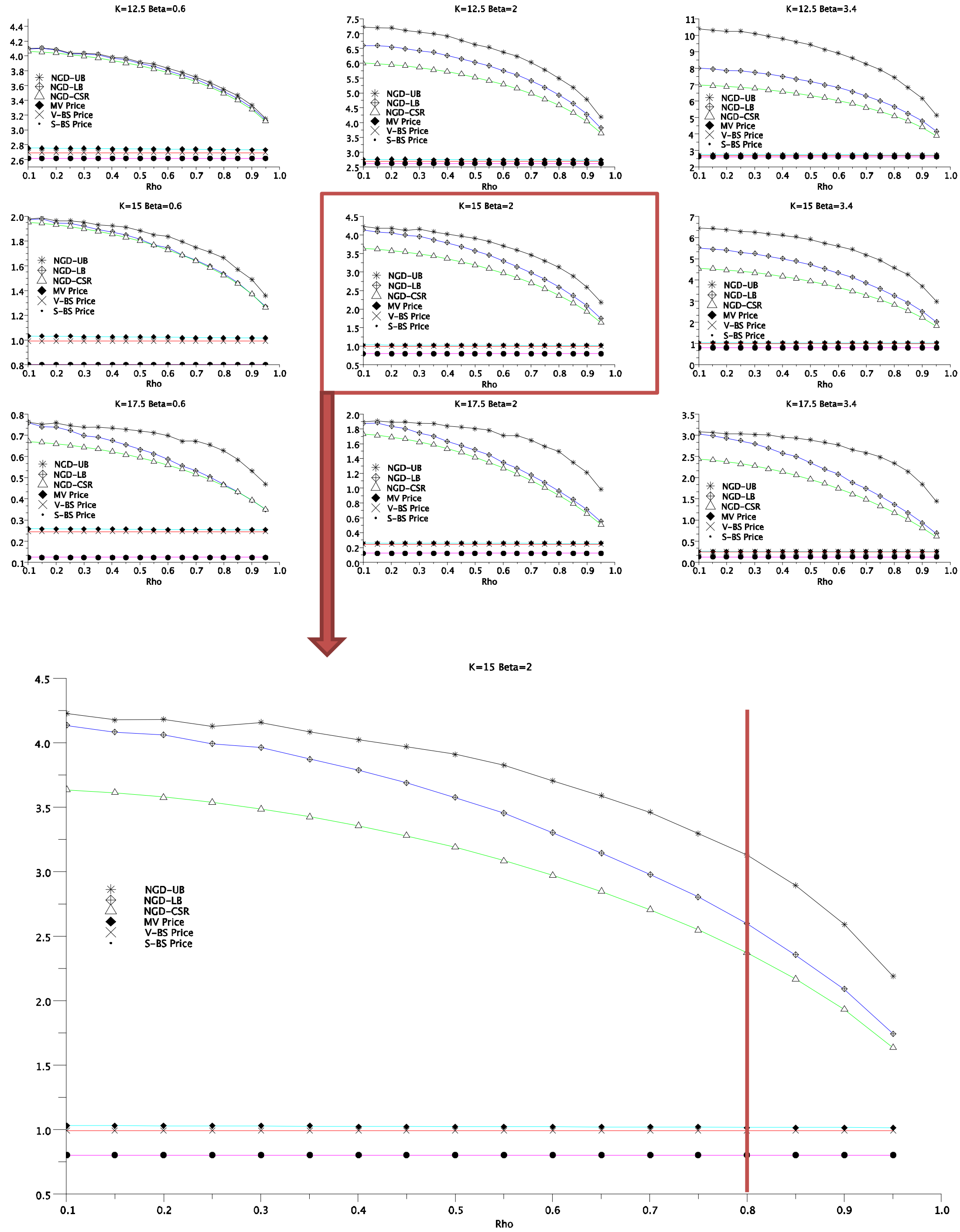

Fig. 1: *Evolution of prices w.r.t.* ρ *for different values of* β *and* K*.*

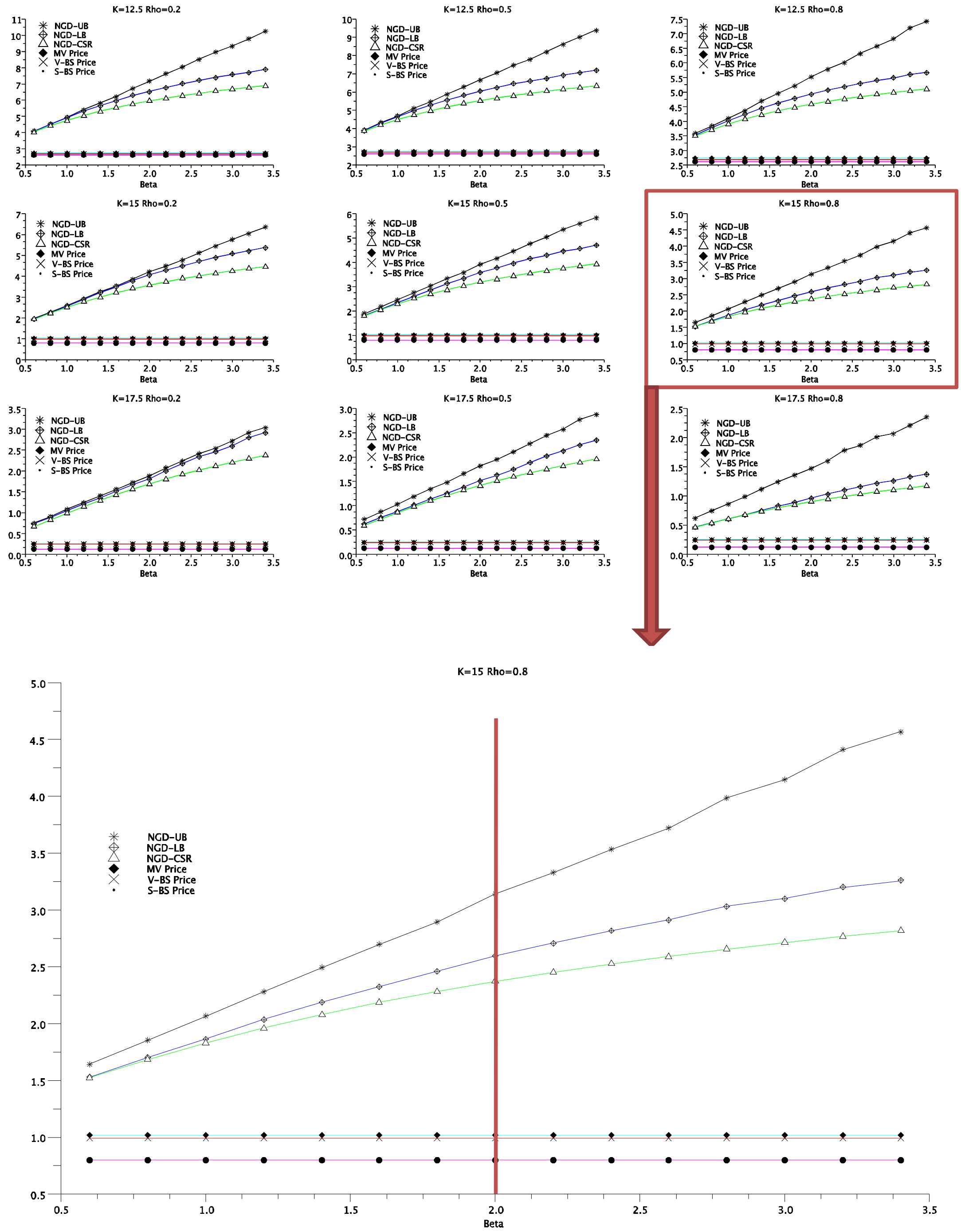

Fig. 2: *Evolution of prices w.r.t. β for different values of ρ and K.*

5 Hedging basis risk

The preceding section allows us to propose a price compatible with the No Good Deal criterium. But as mentioned in Remark 2, there is no natural hedging strategy associated to our No Good Deal criterium. As it is essential, when pricing, to give a hedging strategy, we will study in this section several hedging criteria and compare them numerically. We will first introduce simple strategies : buy and hold and Black Scholes ones. Then, we will consider the hedging criterium introduce by Becherer (2009) and which is related to his NGD definition (see (45)). But as this hedging criterium is not explicit, we will propose a well known, easy to apprehend one : the minimum quadratic error criterium. In order to compare all those criteria, we will compute the value of the associated strategies X_T^{Strat} starting from the previous studied prices: X_0 will be successively equal to the minimum variance price, the CSR price and the mid price of our lower and upper bound that we will call from now NGD price.

5.1 Simple Strategies

We first introduce two buy and hold, naive, strategies :

- Buy and Hold in cash ("BaHCash"): we put all the initial wealth X_0 in cash, thus $X_T^{BaHCash} = X_0 e^{rT}$.
- Buy and Hold in S ("BaHS"): we put all the initial wealth X_0 in the risky tradable asset S, thus $X_T^{BaHS} = X_0 S_T / S_0$.

Then, we consider the Black-Scholes strategy ("BS") linked to the so called "S-BS" price (see (46)). Starting from "S-BS" price at time 0 and following a Black-Scholes strategy we replicate at time T, the payoff $\left(\frac{V_0}{S_0} S_T - K\right)_+$. The difference between the initial wealth X_0 and the price "S-BS" is put in cash. Thus $X_T^{BS} = \left(\frac{V_0}{S_0} S_T - K\right)_+ + (X_0 - S\text{-}BS_0)e^{rT}$.

5.2 Becherer hedging criterium

We have already detail the No Good Deal criterium used by Becherer (2009) (see (45)), which in our simple setup, reduces to CSR condition. We now turn to his hedging condition. His hedging strategy Φ^B is obtained such that the No Good Deal price at time t (here the CSR price, see (43)) is the minimum over all admissible strategies in risky asset Φ^1 of $\rho_t(He^{-rT} - \int_t^T \Phi_s^1 d\tilde{S}_s)$, where ρ_t is an ad hoc risk measure and $\tilde{S}_s = e^{-rs} S_s$, i.e,

$$
\begin{aligned}
p_t^{CSR} &= \operatorname*{ess\,inf}_{\Phi^1} \rho_t \left(He^{-rT} - \int_t^T \Phi_s^1 d\tilde{S}_s \right) \\
&= \rho_t \left(He^{-rT} - \sigma_S \int_t^T e^{-rs} S_s \Phi_s^{1,B} dW_s^0 \right).
\end{aligned}
\tag{47}
$$

This risk measure is not constructed directly from no good deal condition as (40) but rather chosen to be the right risk measure which allows (47) to holds. This risk measure is not explicitly computable but should be obtained numerically as a BSDE solution (see remark 5.0.18.5 in Becherer (2009)). The optimal solution $\Phi^{1,B}$ is also obtain as a BSDE solution but this one can be easily solve in our context : the optimal strategy in the risky asset is given by

$$
\begin{aligned}
\Phi_t^{1,B} &= \frac{\sigma_V}{\sigma_S} e^{-r(T-t)+(\mu_V - \sigma_V h_S \rho + \sigma_V \sqrt{1-\rho^2}\lambda_{max})(T-t)} \frac{V_t}{S_t} \\
&\times \mathcal{N}(d_1(V_t, T-t, K, \mu_V - \sigma_V h_S \rho, \sigma_V)) \left(\rho + h_S \frac{\sqrt{1-\rho^2}}{\lambda_{max}} \right)
\end{aligned}
\tag{48}
$$

Recall that this trading solution is optimal for the CSR price, not for the NGD price. The construction of an ad hoc risk measure, adapted to NGD price, is left to further studies.

But as already seen in the pricing section, the NGD criterium is much more difficult to handle than the condition on the non hedgeable risk premium λ as in Cochrane and Saa-Requejo (2001) and Becherer (2009). So it seems worthy to study an another hedging criterium. As the objective of this paper is to obtain explicit, easy to understand and to implement results, both for the price and the hedging strategy, we choose to study a well known criterium in incomplete market literature : the minimal quadratic error criterium. This criterium and the No Good deal one are connected since they rely on quadratic (but different) criteria. We will see in the numerical application that the result obtained by the Becherer optimal strategy starting from CSR price and the one using the minimal quadric error optimal strategy are very similar.

5.3 Minimal quadratic error hedging

In this section, we present concrete hedging results for the minimal quadratic error criterium. For a given initial wealth X_0, we want to find the self-financed strategy in the *tradable* assets that minimizes the quadratic error (under the historical probability), i.e. the difference between the claim and the final value of the strategy. This concept has been introduced by Föllmer and Sondermann (1986) in the martingale case. It is also study by Duffie and Richardson (1991) and by Schweizer (1992). The general proof was given by Gourieroux et al. (1998). Let $H = (V_T - K)_+$, then H belongs to $L^2(\mathbb{P})$ and mathematically, we want to solve the optimization problem:

$$v(H) \quad := \inf_{(\Phi^0,\Phi^1)\in\mathcal{A}_2} \mathbb{E}\left[H - (\Phi^0_T S^0_T + \Phi^1_T S_T)\right]^2. \tag{49}$$

Via a change of numéraire, we transform the initial problem in order to obtain (local) martingales and perform a projection argument based on the Galtchouk-Kunita-Watanabe theorem. The technical Theorem 6 and all the related tools are given in Appendix section 6.2. The contribution of our proof is the following : the choice of the numéraire U follows directly from the martingale property of the risky assets: we don't need to use the so-called optimal variance measure. The numéraire U the associated change of measure and brownian motions are given by:

$$U_t = e^{-h_S W_t + (r - 3/2h_S^2)t}, \tag{50}$$

$$d\mathbb{Q}^U/d\mathbb{P} = e^{-2h_S W_T - 2h_S^2 T} \tag{51}$$

$$W_t^U = W_t + 2h_S t, \text{ and } W_t^{*,U} = W_t^*. \tag{52}$$

The theorem below gives the explicit solution to the minimal quadratic hedging in the context of Basis Risk.

Theorem 5 *The solution of Problem 49 is given by*

$$\Phi_t^{0,H} = \frac{U_t}{S_t^0}\left[\frac{\sigma_S + h_S}{\sigma_S}\left(X_0 + \int_0^t \left(h_S K_l + \rho\frac{L_l}{U_l}\right) dW_l^U\right) - \frac{1}{\sigma_S}\left(h_S K_t + \rho\frac{L_t}{U_t}\right)\right] \tag{53}$$

and

$$\Phi_t^{1,H} = \frac{U_t}{\sigma_S S_t}\left[\left(h_S K_t + \rho\frac{L_t}{U_t}\right) - h_S\left(X_0 + \int_0^t \left(h_S K_l + \rho\frac{L_l}{U_l}\right) dW_l^U\right)\right]. \tag{54}$$

The minimum is equal to

$$v(H) = e^{(2r - h_S^2)T}\Bigg[\left(e^{-rT} BS(V_0, T, K, \mu_V - \sigma_V h_S \rho, \sigma_V) - X_0\right)^2 + (1 - \rho^2)\mathbb{E}^{\mathbb{Q}^U}\left(\int_0^T \left(\frac{L_t}{U_t}\right)^2 dt\right)\Bigg], \tag{55}$$

where

$$K_t = \frac{e^{-r(T-t)}}{U_t} BS(V_t, T-t, K, \mu_V - \sigma_V h_S \rho, \sigma_V)$$
$$L_t = \sigma_V e^{-r(T-t)+(\mu_V - \sigma_V h_S \rho)(T-t)} V_t \mathcal{N}(d_1(V_t, T-t, K, \mu_V - \sigma_V h_S \rho, \sigma_V)),$$

and BS and d_1 are defined in Equation (56).

Proof See Appendix 6.2.1

Remark 3 *Note that $K_t U_t = e^{-r(T-t)} BS(V_t, T-t, K, \mu_V - \sigma_V h_S \rho, \sigma_V)$ is the Black- Scholes price of a call on V with strike K and maturity T, if the pricing measure is $\mathbb{Q}^0$. This can occur in two contexts : the first one is the minimal variance martingale measure criterium. This is also the case, if V is tradable (i.e. the market is complete) and if $e^{-rt}V_t$ is a $\mathbb{Q}^0$-martingale (which is implied by $\mu_V - \sigma_V h_S \rho = r$). In this case, the process L_t represents the "Delta" of this option.*

If we want to find the minimal initial wealth p_{opt} needed to perform the quadratic hedging, it is clear from Equation (55) that $p_{opt} = e^{-rT} BS(V_0,\, T, K,\, \mu_V - \sigma_V h_S \rho, \sigma_V)$ and of course we get back to the Minimum Variance price p_0^{MV} (see (23)). But as our initial capital is the No Good Deal Price p_0, the optimal quadratic error is higher than the one starting with capital p_0^{MV} : we have an extra term equal to $e^{(2r-h_S^2)T}\left(p_0^{MV} - p_0\right)^2$.

5.4 Numerical Results

In order to quantify the hedging error we adopt three points of view : probability of super-replication, expected loss and Value at Risk.

For the probability of super-replication, we evaluate $\mathbb{P}[X_T^{Strat} \geq (V_T - K)_+]$. It is economically meaningful but has two drawbacks. The first one is theoretical : even if the probability is close to one, the loss might be huge. Moreover, from a numerical point of view, the usual estimator of a probability is very unstable: as it integrates a "one or nothing" function, two close trajectories could lead to significantly different results.

For the expected loss, we compute $\mathbb{E}[\left((V_T - K)_+ - X_T^{Strat}\right)_+]$. This is the classical expected shortfall risk measure. It allows to evaluate the size of the loss, but does not tell how often this loss occurs. From a numerical view point its estimation is more stable.

Both preceding notions are deeply dependent on the level chosen for the initial wealth ("price effect"). If one starts with a significatively higher initial wealth, one will do much better in terms of super-replication and expected losses.

As the NGD prices might be five times higher than the minimum variance one, we expected that both the probability of super-replication and expected losses will perform better starting from NGD prices. To overcome this drawback, we introduce the Value At Risk risk indicator and compute the VaR of the loss at 99%, i.e. the value v such that

$$\mathbb{P}[X_T^{Strat} - (V_T - K)_+ \geq -v] = 99\%.$$

Formally, it is the value we have to add to our strategy to replicate the derivative with a probability equal to 99%. It is also the maximal loss, with probability 99% arising from following the strategy X^{Strat} and delivering the option $(V_T - K)_+$. VaR is a widely used measure of risk.

We plot the results of the simulation in figures 3 and 4 with $\beta = 2$ and $K = 15$. We choose these values because it is reasonable from an economical point of view for β and because the results of simulations do not change a lot for in or out the money derivatives or other values of β.

In order to interpret our numerical results, note that better situations are characterized by probability of super-replication close to 1, small expected loss and low VaR.

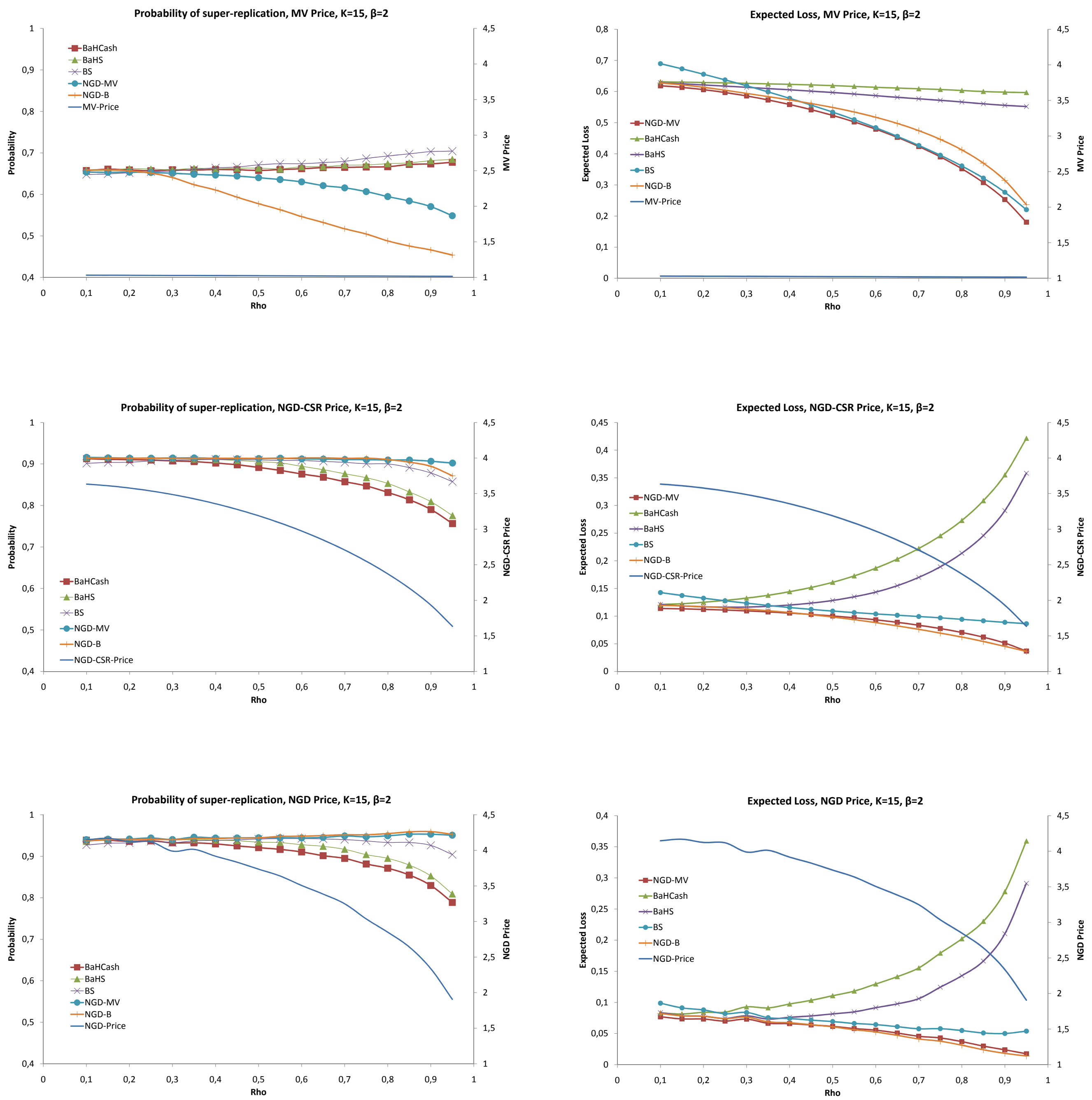

Fig. 3: *Comparison of the probability of super-replication and expected loss for the different strategies starting from "MV-Price", "NGD-CSR" and the middle of "NGD-LB" and "NGD-UB".*

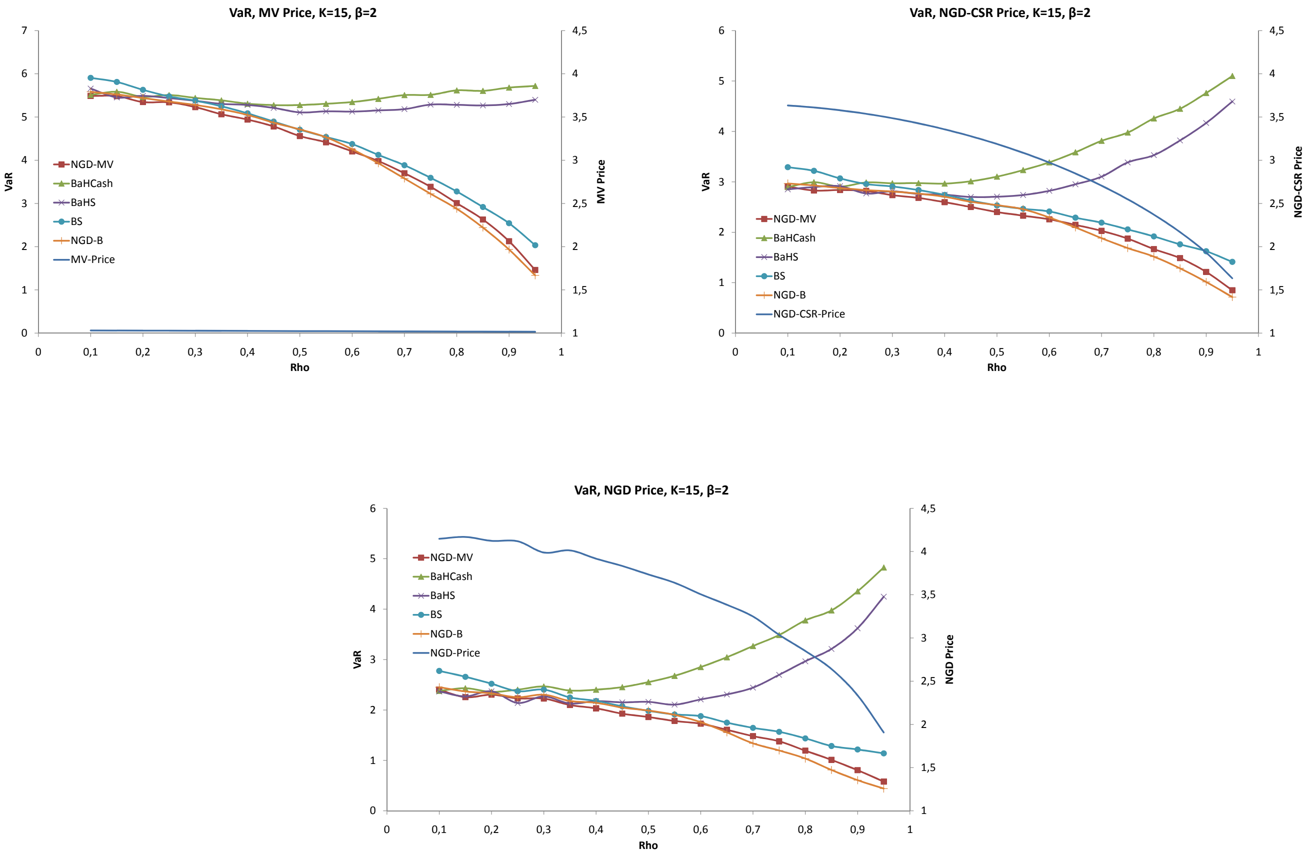

Fig. 4: *Comparison of the value at risk for the different strategies starting from "MV-Price", "NGD-CSR" and the middle of "NGD-LB" and "NGD-UB".*

We can classify our strategies in two categories : the first one contains the "naive" Buy and Hold strategies ("BaHCash" and "BaHS") and the second the more elaborated ones : the strategy of Becherer (2009) ("NGD-B"), the minimum variance strategy "NGD-MV" and the Black-Scholes strategy "BS". We see in figures 3 and 4 that each category have a similar behavior (except for the super-replication criterium).

We first remark that the strategies "NGD-B" and "NGD-MV" have slightly better results than the other ones, especially when the correlation is high, which is satisfying in the context of Basis Risk. Starting from "NGD-CSR" or from "NGD" the probability of super-replication are close to one, the expected loss and the VaR are small. We also remark that the strategy of Becherer (2009) and the minimum variance one have very similar results. This intends to prove that the numerical impact is not significant between those two strategies. Thus, if we want to obtain fully calculable and easy to interpret results, the minimum variance strategy might be a satisfying alternative.

Next, as expected, the results obtained starting from initial prices "NGD-CSR" and "NGD" are very similar : only the level varies. For a correlation of 0.8, the ratio between "NGD" divided by "NGD-CSR" is equal to 1.2 and in average (on the strategy) the probability of super-replication increases of 4%, the expected loss decreases of 35% and the VaR decreases of 19% when starting from "NGD" instead of "NGD-CSR".

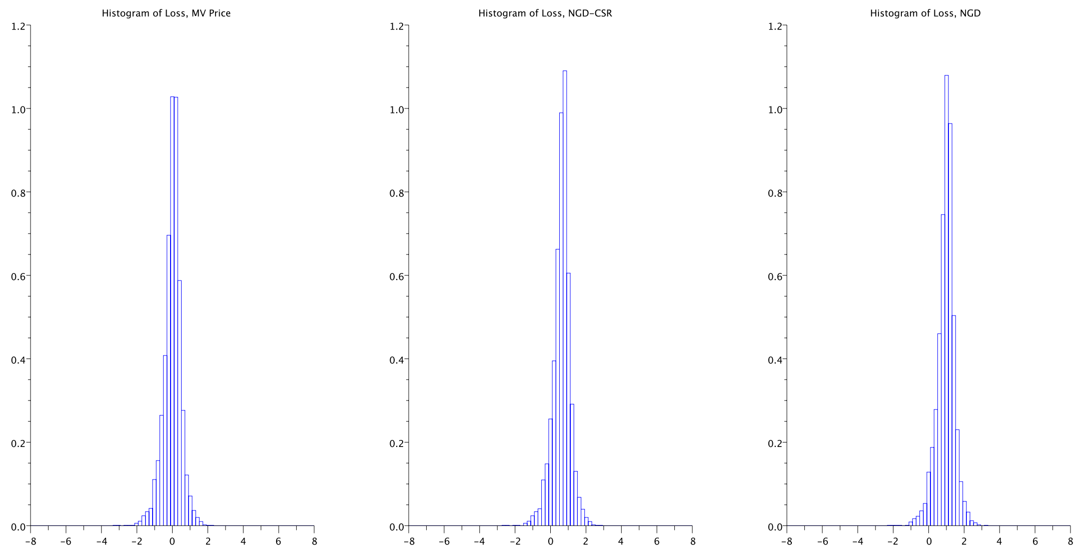

Fig. 5: *Histogram of the loss for "NGD-MV" strategy for the three different prices and* $K = 15$, $\beta = 2$ *and* $\rho = 0.8$

We now observe the dependence in ρ. First when ρ is small all strategies seem to perform similarly : note that this is not true in general choosing another set of parameters. When ρ increases, the prices "NGD-CSR", "NGD-B" and "MV" should decrease (see section 4.5.3). We see that for our set of parameters, "MV" remains almost constant. Now, in the strategies "BaHCash" and "BaHS", the correlation appears only in the initial wealth. Thus starting from X_0 equal to "NGD-CSR" and "NGD" and recalling the definition of our risk measures, it is clear that the probability of super-replication should decrease, the expected loss and the VaR should increase with the ρ. Starting from 'MV" price, the three risk measures should not vary a lot. This is what we observe in figures 3 and 4.

In contrast to buy and hold strategies, "NGD-B", "NGD-MV" and "BS" intend to approach ($a.s.$ for "BS", in a quadratic way for "NGD-MV" and according to the risk measure ρ_t for "NGD-B") the optional call payoff. When ρ increases, both risky assets S and V become similar in term of risk, thus it seems natural that the risk of loss arising from hedging a call written on V with a strategy in S should decrease. Thus we should observe an increase of the probability of super-replication and a decrease of the expected loss and the VaR. Considering figures 3 and 4, we note that this is true for expected loss and VaR but it is less clear for the probability of super-replication. This is even definitively not true for "NGD-B" and "NGD-MV" strategies starting from "MV" price. For "NGD-MV", recall that the minimum variance principle implies to minimize the variance of loss, thus we expect to get a loss which is similar to a Dirac mass in zero : this is confirmed by numerical experiments (see left of figure 5). When evaluating numerically a loss which is similar to a Dirac mass, it is intuitive that the associated probability should be around 1/2 (and the expected loss around 0). Note that starting from a price other than "MV" the distribution of loss is not centered around 0 any more (see figure 5) and we don't have the same numerical problem.

Now, for the probability of super-replication starting from NGD prices, the results are not so clear. Following the "NGD-MV" or the "NGD-B" strategies, the probability seems to be more or less constant and following "BS", it seems to be decreasing. Note that there are still numerical issues associated to the evaluation of a probability which are combined with the "price effect". Finally, we remark that the three sophisticated approaches "NGD-MV", "NGD-B" and "BS" allow to overcome the fact when ρ increases the prices decreases : even if we start with less cash, we perform a better hedging.

6 Appendices

6.1 Technical results

6.1.1 On Black-Scholes formula

We recall the following formula which is analogous to the Black-Scholes formula. All proofs are omitted since they are completely similar to the one of Black-Scholes model which can be found for example in Musiela and Rutkowski (2007) (starting from p.94).
Let Y be a geometric Brownian motion, with drift η and volatility φ, i.e.

$$Y_t = Y_0 \exp\left(\left(\eta - \frac{\varphi^2}{2}\right)t + \varphi W_t\right).$$

Then, the function $BS(Y_t, T-t, K, \eta, \varphi)$ defined by

$$BS(Y_t, T-t, K, \eta, \varphi) = \mathbb{E}[(Y_T - K)_+ \mid \mathcal{F}_t],$$

can be explicitly expressed as

$$BS(Y_t, T-t, K, \eta, \varphi) = Y_t e^{\eta(T-t)}\mathcal{N}(d_1) - K\mathcal{N}(d_0) \tag{56}$$

where[6]

$$d_1 = d_1(Y_t, T-t, K, \eta, \varphi) = \frac{\ln\left(\frac{Y_t}{K}\right) + \left(\eta + \frac{\varphi^2}{2}\right)(T-t)}{\varphi\sqrt{T-t}};\ d_0 = d_0(Y_t, T-t, K, \eta, \varphi) = d_1 - \varphi\sqrt{T-t}$$

By Ito Formula, one can show that

$$dBS(Y_t, T-t, K, \eta, \varphi) = \varphi e^{\eta(T-t)}\mathcal{N}(d_1)Y_t dW_t. \tag{57}$$

$$\frac{\partial BS}{\partial \eta}(Y_t, T-t, K, \eta, \varphi) = (T-t)K\mathcal{N}(d_0) > 0, \tag{58}$$

The last equation implies that the Black-Scholes function BS is increasing w.r.t. η.

6.1.2 Proof of Lemma 1

Proof Let $\lambda \in \Lambda$, since Z_T^λ is a martingale, $\mathbb{E}Z_T^\lambda = Z_0^\lambda = 1$. We define the following process

$$\bar{Z}_t^\lambda = \exp\left(-2h_S W_t - 2h_S^2 t + 2\int_0^t \lambda_s dW_s^\star - 2\int_0^t \lambda_s^2 ds\right).$$

$\bar{Z}^\lambda$ is a Doléans-Dade process and thus a continuous local martingale (see Karatzas and Shreve (1991), p.191). We are going to show that $\bar{Z}^\lambda$ is a martingale.
We clearly have that $\bar{Z}_t^\lambda \le \left(Z_t^\lambda\right)^2$, $\forall t \in [0,T]$. Since Z^λ is a square integrable martingale, the Doob maximal inequality (see Karatzas and Shreve (1991), Theorem 1.3.8 p.14) implies that

$$\mathbb{E}\left(\sup_{t\in[0,T]} Z_t^\lambda\right)^2 \le 4\mathbb{E}[\left(Z_T^\lambda\right)^2] < +\infty.$$

[6] $\mathcal{N}$ is the normal distribution function, i.e. $\mathcal{N}(d) = \int_{-\infty}^d \frac{e^{-x^2/2}}{\sqrt{2\pi}}dx$

Let τ be a stopping time such that $\mathbb{P}(0 \le \tau \le T) = 1$, then $\bar{Z}^\lambda_\tau \le \sup_{t\in[0,T]} \bar{Z}^\lambda_t$. So we deduce that

$$\mathbb{E}\left[\sup_{\tau\in[0,T]} \bar{Z}^\lambda_\tau\right] \le \mathbb{E}\left[\sup_{t\in[0,T]} \bar{Z}^\lambda_t\right] \le \mathbb{E}\left[\sup_{t\in[0,T]} \left(Z^\lambda_t\right)^2\right] = \mathbb{E}\left[\left(\sup_{t\in[0,T]} Z^\lambda_t\right)^2\right] < +\infty.$$

Thus $\bar{Z}^\lambda$ is a continuous local martingale of class (DL) (see Karatzas and Shreve (1991), definition 1.4.8 p.24). This shows that $\bar{Z}^\lambda$ is a martingale (see Karatzas and Shreve (1991), problem 1.5.19 (i) p.36) and thus $\mathbb{E}\left(\bar{Z}^\lambda_T\right) = 1$. Then, we can define the following probability measure

$$d\tilde{\mathbb{Q}}/d\mathbb{P} = \bar{Z}^\lambda_T.$$

Using Bayes Formula

$$\mathbb{E}((Z^\lambda_T)^2) = \mathbb{E}^{\tilde{\mathbb{Q}}}\left(e^{h_S^2 T + \int_0^T \lambda_s^2 ds}\right) \text{ thus, } \operatorname{Var}(Z^\lambda_T) = e^{h_S^2 T}\mathbb{E}^{\tilde{\mathbb{Q}}}\left(e^{\int_0^T \lambda_s^2 ds}\right) - 1.$$

6.1.3 Proof of Lemma 2

We will prove the following lemma for ease of exposure. Let $X \in L^2$, $X \ge 0$ such that $X\mathbf{1}_{X>0}$ has a density with respect to Lebesgue measure and γ a positive number.

$$p_0^{opt} = \sup_{\substack{Y\ge 0,\ \mathbb{E}Y=1\\ \mathbb{E}Y^2 \le 1+\gamma^2}} \mathbb{E}[YX] \tag{59}$$

Lemma 3 *The solution of Problem 59 is :*
if $1 - \gamma\frac{\mathbb{E}(X)}{\sqrt{\operatorname{Var} X}} \ge 0$ *then* $Y^{opt} = 1 + \gamma\frac{X-\mathbb{E}(X)}{\sqrt{\operatorname{Var} X}}$ *and* $p_0^{opt} = \mathbb{E}(X) + \gamma\sqrt{\operatorname{Var} X}$.
if $1 - \gamma\frac{\mathbb{E}(X)}{\sqrt{\operatorname{Var} X}} < 0$ *then* $Y^{opt} = \frac{(X-\alpha)_+}{\mathbb{E}(X-\alpha)_+}$ *and* $p_0^{opt} = \alpha + (1+\gamma)^2\mathbb{E}(X-\alpha)_+$, *where there exists* α, *a positive number, such that*[7]

$$\frac{\mathbb{E}(X-\alpha)_+^2}{\mathbb{E}^2(X-\alpha)_+} = 1+\gamma^2. \tag{60}$$

Proof If $1 - \gamma\frac{\mathbb{E}(X)}{\sqrt{\operatorname{Var} X}} \ge 0$, then it is straightforward that $\mathbb{E}Y^{opt}X = \mathbb{E}(X) + \gamma\sqrt{\operatorname{Var} X}$. Let Y such that $Y \ge 0$, $\mathbb{E}Y = 1$ and $\mathbb{E}Y^2 \le 1+\gamma^2$ then $\operatorname{Var} Y \le \gamma^2$ and by Cauchy-Schwartz inequality

$$\mathbb{E}(YX)\mathbb{E}((Y-\mathbb{E}Y)(X-\mathbb{E}X)) + \mathbb{E}X \le \sqrt{\operatorname{Var} X}\sqrt{\operatorname{Var} Y} + \mathbb{E}X \le \gamma\sqrt{\operatorname{Var} X} + \mathbb{E}X = \mathbb{E}[Y^{opt}X].$$

To prove that Y^{opt} is the optimal solution of (59), it remains to verify that it satisfies the constraints. $Y^{opt} = 1 - \gamma\frac{\mathbb{E}X}{\sqrt{\operatorname{Var} X}} + \gamma\frac{X}{\sqrt{\operatorname{Var} X}} \ge 0$ by assumption (recall that $X \ge 0$). The two others constraints are straightforward.
If $1 - \gamma\frac{\mathbb{E}(X)}{\sqrt{\operatorname{Var} X}} < 0$, assume that there exists α such that condition (60) is satisfied. Then it is straightforward that

$$\mathbb{E}\left[Y^{opt}X\right] = \frac{\mathbb{E}[(X-\alpha)_+(X-\alpha+\alpha)]}{\mathbb{E}(X-\alpha)_+} = \alpha + \frac{\mathbb{E}(X-\alpha)_+^2}{\mathbb{E}(X-\alpha)_+} = \alpha + (1+\gamma^2)\mathbb{E}(X-\alpha)_+$$

using condition 60. Let Y such that $Y \ge 0$, $\mathbb{E}Y = 1$ and $\mathbb{E}Y^2 \le 1+\gamma^2$ then by Cauchy-Schwartz inequality and condition (60)

$$\begin{aligned}\mathbb{E}[YX] &= \mathbb{E}\left(Y(X-\alpha)_+\right) + \alpha + \mathbb{E}\left(Y(X-\alpha)\mathbf{1}_{X<\alpha}\right) \le \sqrt{\mathbb{E}(X-\alpha)_+^2}\sqrt{\mathbb{E}Y^2} + \alpha \\ &\le \sqrt{1+\gamma^2}\sqrt{1+\gamma^2}\mathbb{E}(X-\alpha)_+ + \alpha = \mathbb{E}[Y^{opt}X].\end{aligned}$$

[7] The term $\mathbb{E}^2[A]$ denotes $(\mathbb{E}[A])^2$

Y^{opt} is thus optimal solution for 59 because that it satisfies the constraints (see condition 60). It remains to prove that there exists some α such that condition 60 is satisfied. Let $f(x) = \frac{\mathbb{E}(X-x)_+^2}{\mathbb{E}^2(X-x)_+}$ then $f(0) = \frac{\operatorname{Var} X}{\mathbb{E}^2 X} + 1 < 1+\gamma^2$ by assumption. Below we show that there exists $\alpha_0 > 0$ such that $f(\alpha_0) \geq 1+\gamma^2$, thus by continuity of f there will exist some $\alpha > 0$ such that $f(\alpha) = 1+\gamma^2$. We prove first that there exist α_0 such that $\mathbb{P}(X > \alpha_0) = \frac{1}{1+\gamma^2}$. Such an α_0 exists because

$$
\begin{aligned}
1 - \gamma \frac{\mathbb{E}(X)}{\sqrt{\operatorname{Var} X}} < 0 \Leftrightarrow \operatorname{Var}[X \mathbf{1}_{X>0}] < \gamma^2 \mathbb{E}^2[X\mathbf{1}_{X>0}] \Leftrightarrow \mathbb{E}[X^2 \mathbf{1}_{X>0}] < (\gamma^2+1)\mathbb{E}\left[X\left(\sqrt{\mathbf{1}_{X>0}}\right)^2\right] \\
\Rightarrow \mathbb{E}[X^2\mathbf{1}_{X>0}] < (\gamma^2+1)\mathbb{E}\left[X^2\mathbf{1}_{X>0}\right]\mathbb{P}(X>0),
\end{aligned}
$$

by Cauchy Schwartz inequality. Thus $\mathbb{P}(X=0) \leq \frac{\gamma^2}{1+\gamma^2}$ and by continuity of $x \to \mathbb{P}(X \leq x)$, for $x > 0$ there exists α_0 such that $\mathbb{P}(X \leq \alpha_0) = \frac{\gamma^2}{1+\gamma^2}$. Then by Cauchy Schwartz inequality,

$$
\begin{aligned}
\mathbb{E}^2(X-\alpha_0)_+ &= \mathbb{E}^2(X\mathbf{1}_{X>\alpha_0}) - 2\alpha_0\mathbb{P}(X>\alpha_0)\mathbb{E}(X\mathbf{1}_{X>\alpha_0}) + \alpha_0^2\mathbb{P}^2(X>\alpha_0) \\
&\leq \mathbb{P}(X>\alpha_0)\mathbb{E}(X^2\mathbf{1}_{X>\alpha_0}) - 2\alpha_0\mathbb{P}(X>\alpha_0)\mathbb{E}(X\mathbf{1}_{X>\alpha_0}) + \alpha_0^2\mathbb{P}^2(X>\alpha_0) \\
&\leq \frac{1}{1+\gamma^2}\left(\mathbb{E}X^2\mathbf{1}_{X>\alpha_0} - 2\alpha_0\mathbb{E}X\mathbf{1}_{X>\alpha_0} + \alpha_0^2\mathbb{P}(X>\alpha_0)\right) \\
&\leq \frac{1}{1+\gamma^2}\mathbb{E}(X-\alpha_0)_+^2.
\end{aligned}
$$

Thus $f(\alpha_0) \geq 1+\gamma^2$ which concludes the proof.

6.2 Results on Mean Variance Hedging

In a first time, we will consider the case of a general contingent claim H. Of course, when we study quadratic hedging we have to assume that:

Assumption 2 *The contingent claim H belongs to $L^2(\mathbb{P})$.*

Mathematically, we want to solve the optimization problem 49. The first question is whether this problem admits a solution or not? The answer is yes and we will construct it explicitly. In fact, from the definition of $\mathcal{A}_2$, we can see directly that the solution exists. It is well known that $L^2(\mathbb{P})$ is an Hilbert space under the inner product $(.|.)$ defined by $(X|Y) = \mathbb{E}(XY)$ and the associated norm $\|.\|$. The set $\{\Phi_T^0 S_T^0 + \Phi_T^1 S_T \,|(\Phi^0,\Phi^1) \in \mathcal{A}_2\}$ is a linear closed subset of $L^2(\mathbb{P})$ (see Delbaen and Schachermayer (1996) Thm. 2.2.) Thus Problem (49) admits a solution by an Hilbert space projection Theorem (see for example Luenberger (1969)). The natural ideal followed by Duffie and Richardson (1991) and later by Schweizer (1992) is to use orthogonality and say that Φ^{0^*} and Φ^{1^*} are solutions of Problem 49 if and only for any $(\Phi^0,\Phi^1) \in \mathcal{A}_2$, $\left(H - ({\Phi^0}^*_T S_T^0 + {\Phi^1}^*_T S_T)|\Phi_T^0 S_T^0 + \Phi_T^1 S_T\right) = 0$. This leads to a PDE (see Equation 3.1 in Schweizer (1992) for example) which is not straightforward to solve explicitly. The other natural idea is to use a projection argument and to get the explicit projection of H on S^0 and S. But as S^0 and S are not martingale this is not technically possible. So we follow the idea of Gourieroux et al. (1998) and transform the initial problem in order to get (local) martingales and achieve the projection argument.

Assumption 3 *We assume that there exists a numéraire U, i.e., an $\mathbb{F}$-adapted, positive semi-martingale such that $1/U$ is also a semi-martingale and $U_0 = 1$, satisfying $U_T \in L^2(\mathbb{P})$.*

We first rewrite problem 49 using from self-financing property with respect to U. A strategy $(\Phi^0,\Phi^1) \in \mathcal{S}$ is U-self-financing in $\left(\frac{S^0}{U}, \frac{S}{U}\right)$ if and only if : $(\Phi^0,\Phi^1) \in L\left(\frac{S^0}{U}, \frac{S}{U}\right)$ and $\Phi^0 d\left(\frac{S^0}{U}\right) + \Phi^1 d\left(\frac{S}{U}\right) = d\left(\frac{\Phi^0 S^0 + \Phi^1 S}{U}\right)$. We call $\mathcal{A}_2^U$ the set of such strategies. Then if $(\Phi^0,\Phi^1) \in \mathcal{A}_2^U$, we get

$$
X_t = U_t\left(X_0 + \int_0^t \Phi_s^0 d\frac{S_s^0}{U_s} + \int_0^t \Phi_s^1 d\frac{S_s}{U_s}\right).
$$

Under Assumption 3, $\mathcal{A}_2 = \mathcal{A}_2^U$: there is equivalence between different notions of self-financing. The proof is this result is very similar to Proposition 3.2. in Gourieroux et al. (1998) and omitted. Now, we can rewrite our problem as follows :

$$\begin{aligned} v(H) &= \inf_{(\Phi^0,\Phi^1)\in\mathcal{A}_2^U} \mathbb{E}\left[U_T^2\left(\frac{H}{U_T} - \left(X_0 + \int_0^T \left(\Phi_t^0 d\frac{S_t^0}{U_t} + \Phi_t^1 d\frac{S_t}{U_t}\right)\right)\right)^2\right] \\ &= \inf_{(\Phi^0,\Phi^1)\in\mathcal{A}_2^U} \mathbb{E}(U_T^2)\mathbb{E}^{\mathbb{Q}^U}\left[\frac{H}{U_T} - \left(X_0 + \int_0^T \left(\Phi_t^0 d\frac{S_t^0}{U_t} + \Phi_t^1 d\frac{S_t}{U_t}\right)\right)\right]^2, \end{aligned}$$

where the probability $\mathbb{Q}^U$ is defined by $d\mathbb{Q}^U/d\mathbb{P} = U_T^2/\mathbb{E}(U_T^2)$. The idea is to find the right U such that $\frac{S^0}{U}$ and $\frac{S}{U}$ are $\mathbb{Q}^U$ (local) martingale and thus be able to do the projection of $\frac{H}{U_T}$ on $\frac{S^0}{U}$ and $\frac{S}{U}$ (by Galtchouk-Kunita-Watanabe Projection Theorem, see for example Jacod (1979)). Contrary to Gourieroux et al. (1998), we do not introduce the so-called variance-optimal martingale measure in order to solve our problem but we show directly that the fact that $\frac{S^0}{U}$ and $\frac{S}{U}$ are $\mathbb{Q}^U$ (local) martingale imposes a particular form on U (see Lemma 4 below). We then solve the problem using Galtchouk-Kunita-Watanabe Projection Theorem for a general H (see Theorem 6) and Ito calculus for a call option (see Theorem 5).

Lemma 4 *Let U such Assumption 3 holds. We further assume that* $\ln(U)$ *is an Ito process, i.e. there exist progressively measurable processes a, λ in $L^2_{loc}((W,W^*))$ and $c \in L^1([0,T])$ such that*

$$dU_t^\lambda = U_t^\lambda(a_t dW_t + \lambda_t dW_t^* + c_t dt).$$

Then, $\frac{S^0}{U^\lambda}$ and $\frac{S}{U^\lambda}$ are local martingale under the measure $\mathbb{Q}^{U^\lambda}$ defined by $d\mathbb{Q}^{U^\lambda}/d\mathbb{P} = \left(U_T^\lambda\right)^2/\mathbb{E}\left(U_T^\lambda\right)^2$ if and only if $a_t = -h_S$ and $c_t = r - \lambda_t^2 - h_S^2$, i.e.

$$U_t^\lambda = e^{-h_S W_t + \int_0^t \lambda_s dW_s^* - 3/2(h_S^2 t + \int_0^t \lambda_s^2 ds) + rt}. \tag{61}$$

If λ is deterministic, then $\frac{S^0}{U^\lambda}$ and $\frac{S}{U^\lambda}$ are $\mathbb{Q}^{U^\lambda}$-martingale.

Proof To show that $\frac{S^0}{U^\lambda}$ and $\frac{S}{U^\lambda}$ are local martingales under the measure $\mathbb{Q}^{U^\lambda}$, we are going to compute the stochastic differential equation satisfied by these processes and see under which conditions they have no drift term. We set two processes W^{U^λ} and W^{*,U^λ} which, thanks to Girsanov Theorem will be Brownian motions under the probability $\mathbb{Q}^{U^\lambda}$:

$$W_t^{U^\lambda} = W_t - 2a_t t,\; W_t^{U^\lambda,*} = W_t^* - 2\lambda_t t. \tag{62}$$

Then, the processes U^λ, S, and S^0 satisfies:

$$\begin{aligned} dU_t^\lambda &= U_t^\lambda\left(a_t dW_t^{U^\lambda} + \lambda_t dW_t^{U^\lambda,*} + (c_t + 2a_t^2 + 2\lambda_t^2)dt\right) \\ dS_t &= S_t\left((r + \sigma_S h_S + 2a_t\sigma_S)dt + \sigma_S dW_t^{U^\lambda}\right) \\ dS_t^0 &= rS_t^0 dt \end{aligned}$$

Thus, by Ito formula applied to $f(x,y) = \frac{x}{y}$, we have

$$d\frac{S_t^0}{U_t^\lambda} = r\frac{S_t^0}{U_t^\lambda}dt - \frac{S_t^0}{{U_t^\lambda}^2}dU_t^\lambda + \frac{S_t^0}{{U_t^\lambda}^3}d < U^\lambda >_t = \frac{S_t^0}{U_t^\lambda}\left[(r - c_t - a_t^2 - \lambda_t^2)dt - a_t dW_t^{U^\lambda} - \lambda_t dW_t^{U^\lambda,*}\right],$$

and

$$\begin{aligned} d\frac{S_t}{U_t^\lambda} &= \frac{dS_t}{U_t^\lambda} - \frac{S_t}{{U_t^\lambda}^2}dU_t^\lambda + \frac{S_t}{{U_t^\lambda}^3}d < U^\lambda >_t - \frac{1}{{U_t^\lambda}^2}d < S, U^\lambda >_t \\ &= \frac{S_t}{U_t}\left[(r + \sigma_S h_S + a_t\sigma_S - c_t - a_t^2 - \lambda_t^2)dt + (\sigma_S - a_t)dW_t^U - \lambda_t dW_t^{U,*}\right], \end{aligned}$$

Thus, these processes are local martingale if and only if

$$r + \sigma_S h_S + a_t \sigma_S - c_t - a_t^2 - \lambda_t^2 = 0 \text{ and } r - c_t - a_t^2 - \lambda_t^2 = 0 \tag{63}$$

$$\int_0^T \left(\frac{S_t^0}{U_t^\lambda}\right)^2 dt < \infty \text{ and } \int_0^T \left(\frac{S_t}{U_t^\lambda}\right)^2 dt < \infty. \tag{64}$$

The inequalities in (64) hold true because $\frac{S^0}{U^\lambda}$ and $\frac{S}{U^\lambda}$ are continuous. The unique solution of this system (63) is $c_t = r - \lambda_t^2 - h_S^2$ and $a_t = -h_S$. With these parameters, the process U_t^λ is the same as those described by (61). We also get that

$$d\left(\frac{S_t^0}{U_t^\lambda}\right) = \frac{S_t^0}{U_t^\lambda}\left[h_S dW_t^{U^\lambda} - \lambda_t dW_t^{U^\lambda,*}\right], \tag{65}$$

$$d\left(\frac{S_t}{U_t^\lambda}\right) = \frac{S_t}{U_t}\Big[(\sigma_S + h_S) dW_t^{U^\lambda} - \lambda_t dW_t^{U^\lambda,*}\Big]. \tag{66}$$

Note that if λ is such that

$$\mathbb{E}^{\mathbb{Q}^{U^\lambda}} \int_0^T \left(\frac{S_t^0}{U_t^\lambda}\right)^2 dt < \infty \text{ and } \mathbb{E}^{\mathbb{Q}^{U^\lambda}} \int_0^T \left(\frac{S_t}{U_t^\lambda}\right)^2 dt < \infty,$$

$\frac{S^0}{U^\lambda}$ and $\frac{S}{U^\lambda}$ are $\mathbb{Q}^{U^\lambda}$ martingale (see Musiela and Rutkowski (2007) p571). This is for example the case with deterministic λ.

From Lemma 4, we get an explicit form for the numéraire U^λ but there are still a lot of possible choices. In a first time, we can restricted our attention to constant process λ : this allows us to compute $\mathbb{E}[U_T^\lambda]$ and $\mathbb{E}[{U_T^\lambda}^2]$. We choose to use the particular numéraire U^0 and thus solve

$$v(H) = \inf_{(\Phi^0,\Phi^1)\in\mathcal{A}_2^{U^0}} \mathbb{E}(\left(U_T^0\right)^2)\mathbb{E}^{\mathbb{Q}^{U^0}}\left[\frac{H}{U_T^0} - \left(X_0 + \int_0^T (\Phi_t^0 d\frac{S_t^0}{U_t^0} + \Phi_t^1 d\frac{S_t}{U_t^0})\right)\right]^2 \tag{67}$$

Two reasons motivate this choice. The first one is a financial argument : going back to Equation (61), the only process U^λ which is replicable from the tradeable assets (i.e. which does not depends on W^*), and thus can be called a numéraire from a financial point of view is U^0. The second reason is the mathematical tractability, see Remark 4.
From now on, we will write U for U^0, recall equations (50), (51) and (52). The following Theorem gives the solution to Problem (49) for general H.

Theorem 6 *Assume that Assumption 2 holds. Consider the following Galtchouk-Kunita-Watanabe decomposition* $(\Phi^{0,H}, \Phi^{1,H}, b)$ *for* $0 \le t \le T$

$$\mathbb{E}^{\mathbb{Q}^U}\left(\frac{H}{U_T}|\mathcal{F}_t\right) = \mathbb{E}^{\mathbb{Q}^U}\left(\frac{H}{U_T}\right) + \int_0^t \Phi_l^{0,H} d\frac{S_l^0}{U_l} + \int_0^t \Phi_l^{1,H} d\frac{S_l}{U_l} + \int_0^t b_l dW_l^{*,U}. \tag{68}$$

Then Problem (49) is equivalent to

$$\begin{aligned}\inf_{(\Phi^0,\Phi^1)\in\mathcal{A}_2^U} \mathbb{E}(U_T^2)&\left[\left[\mathbb{E}^{\mathbb{Q}^U}\left(\frac{H}{U_T}\right) - X_0\right]^2 + \mathbb{E}^{\mathbb{Q}^U}\left(\int_0^T b_t^2 dt\right) + \right.\\ &\left.\mathbb{E}^{\mathbb{Q}^U}\left(\int_0^T \left(h_S(\Phi_t^{0,H} - \Phi_t^0)\frac{S_t^0}{U_t} + (\sigma_S + h_S)(\Phi_t^{1,H} - \Phi_t^1)\frac{S_t}{U_t}\right)^2 dt\right)\right]\end{aligned} \tag{69}$$

If $(\Phi^{0,H}, \Phi^{1,H}) \in \mathcal{A}_2^U$*, then* $\Phi^0 = \Phi^{0,H}$ *and* $\Phi^1 = \Phi^{1,H}$ *are solutions of Problem 49. The minimum is equal to*

$$v(H) = e^{(2r - h_S^2)T}\left[\left(\mathbb{E}^{\mathbb{Q}^U}\left(\frac{H}{U_T}\right) - X_0\right)^2 + \mathbb{E}^{\mathbb{Q}^U}\left(\int_0^T b_t^2 dt\right)\right].$$

Proof We have to solve (67). Let $K_t = \mathbb{E}^{\mathbb{Q}^U}\left(\frac{H}{U_T}|\mathcal{F}_t\right)$, using Galtchouk-Kunita-Watanabe decomposition on the $\mathbb{Q}^U$ martingale K under Assumption 2, we get that

$$K_t = \mathbb{E}^{\mathbb{Q}^U}\left(\frac{H}{U_T}\right) + \int_0^t \Phi_l^{0,H} d\frac{S_l^0}{U_l} + \int_0^t \Phi_l^{1,H} d\frac{S_l}{U_l} + R_t^H, \quad 0 \le t \le T,$$

where R^H is a L^2-martingale orthogonal to $\frac{S^0}{U}$ and $\frac{S}{U}$, i.e. $< R_t^H, \frac{S_t^0}{U_t} >= 0$ and $< R_t^H, \frac{S_t}{U_t} >= 0$. Thus as $K_T = \frac{H}{U_T}$ problem (67) can be rewrite as

$$\begin{aligned}
v(H) = &\inf_{(\Phi^0,\Phi)\in\mathcal{A}_2^U} \mathbb{E}(U_T^2)\mathbb{E}^{\mathbb{Q}^U}\Bigg[\mathbb{E}^{\mathbb{Q}^U}\left(\frac{H}{U_T}\right) - X_0 + \int_0^T (\Phi_t^{0,H} - \Phi_t^0) d\frac{S_t^0}{U_t} \\
&+ \int_0^T (\Phi_t^{1,H} - \Phi_t^1) d\frac{S_t}{U_t} + R_T^H\Bigg]^2 \\
= &\inf_{(\Phi^0,\Phi)\in\mathcal{A}_2^U} \mathbb{E}(U_T^2)\Bigg[\left(\mathbb{E}^{\mathbb{Q}^U}\left(\frac{H}{U_T}\right) - X_0\right)^2 + \left(R_T^H\right)^2 \\
&+ \mathbb{E}^{\mathbb{Q}^U}\left[\int_0^T (\Phi_t^{0,H} - \Phi_t^0) d\frac{S_t^0}{U_t} + \int_0^T (\Phi_t^{1,H} - \Phi_t^1) d\frac{S_t}{U_t}\right]^2\Bigg].
\end{aligned} \tag{70}$$

As $\frac{S^0}{U}$ and $\frac{S}{U}$ are not orthogonal, we can not continue directly the computation. We have to decompose this two processes on W^U and $W^{*,U}$ which are orthogonal (see equation (62) for definition of those processes). Since R^H is a square integrable martingale, the Theorem of Martingale representation (see for example D. Revuz and M. Yor) asserts that there exists some progressively measurable processes a and b such that $\mathbb{E}\int_0^T a_t^2 dt < +\infty$ and $\int_0^T |b_t| dt < +\infty$:

$$R_t^H = \int_0^t a_l dW_l^U + \int_0^t b_l dW_l^{*,U}$$

Recalling equation (65) and (66) with $\lambda = 0$, the orthogonality conditions lead to

$$a_t \frac{S_t^0}{U_t} h_S = 0 \text{ and } a_t \frac{S_t}{U_t}(h_S + \sigma_S) = 0.$$

Thus $a_t = 0$ and $R_t^H = \int_0^t b_l dW_l^{*,U}$. Replacing R^H in equation (70) and using again (65) and (66) with $\lambda = 0$

$$\begin{aligned}
v(H) = &\inf_{(\Phi^0,\Phi^1)\in\mathcal{A}_2^U} \mathbb{E}(U_T^2)\Bigg[\left[\mathbb{E}^{\mathbb{Q}^U}\left(\frac{H}{U_T}\right) - X_0\right]^2 + \mathbb{E}^{\mathbb{Q}^U}\left(\int_0^T b_t dW_t^{*,U}\right)^2 + \\
&\mathbb{E}^{\mathbb{Q}^U}\left(\int_0^T \left(h_S(\Phi_t^{0,H} - \Phi_t^0)\frac{S_t^0}{U_t} + (\sigma_S + h_S)(\Phi_t^{1,H} - \Phi_t^1)\frac{S_t}{U_t}\right) dW_t^U\right)^2\Bigg] \\
= &\inf_{(\Phi^0,\Phi^1)\in\mathcal{A}_2^U} \mathbb{E}(U_T^2)\Bigg[\left[\mathbb{E}^{\mathbb{Q}^U}\left(\frac{H}{U_T}\right) - X_0\right]^2 + \mathbb{E}^{\mathbb{Q}^U}\left(\int_0^T b_t^2 dt\right) + \\
&\mathbb{E}^{\mathbb{Q}^U}\left(\int_0^T \left(h_S(\Phi_t^{0,H} - \Phi_t^0)\frac{S_t^0}{U_t} + (\sigma_S + h_S)(\Phi_t^{1,H} - \Phi_t^1)\frac{S_t}{U_t}\right)^2 dt\right)\Bigg]
\end{aligned}$$

The minimum is clearly obtain for $(\Phi^0, \Phi^1) \in \mathcal{A}_2^U$ such that $h_S(\Phi_t^{0,H} - \Phi_t^0)\frac{S_t^0}{U_t} + (\sigma_S + h_S)(\Phi_t^{1,H} - \Phi_t^1)\frac{S_t}{U_t} = 0$ $\mathbb{Q}^U - p.s.$ If $(\Phi^{0,H}, \Phi^{1,H}) \in \mathcal{A}_2^U$, then $\Phi^0 = \Phi^{0,H}$ and $\Phi^1 = \Phi^{1,H}$ are solutions of Problem 49.

If we want to hedge some practical examples of derivative H, we have to perform the Galtchouk-Kunita-Watanabe of $\mathbb{E}^{\mathbb{Q}^U}\left(\frac{H}{U_T}|\mathcal{F}_t\right)$ and find explicitly $(\Phi^{0,H}, \Phi^{1,H}, b)$. This will be done by Ito Formula. We will compute explicitly the solution for a call option on the non-traded asset, i.e. $H = (V_T - K)_+$, in Theorem 5.

Remark 4 *If we choose to solve Problem 67 with $\lambda \neq 0$ instead of $\lambda = 0$, we are not able to find so easily a self-financing strategy, which achieves the minimum. In fact when $\lambda \neq 0$, in Problem 69 the strategy (Φ^0, Φ^1) also appears in the second term. So if for minimizing we put to zero both integrals, we get two equations and thus a unique strategy as a solution. Unfortunately, this strategy is not self-financed. Thus we have to introduce the self-financing constraints and then minimize the sum of the integrals (and not put each of them to zero). This problem is not mathematically tractable.*

6.2.1 Proof of Theorem 5

Proof Using the results of Theorem 6, it is sufficient to compute the Galtchouk- Kunita-Watanabe decomposition of the process K_t. We first remark that K can be rewritten using Bayes Formula as

$$K_t = \mathbb{E}^{\mathbb{Q}^U}\left(\frac{(V_T-K)_+}{U_T}|\mathcal{F}_t\right) = \frac{\mathbb{E}\left(U_T(V_T-K)_+|\mathcal{F}_t\right)}{\mathbb{E}\left({U_T}^2|\mathcal{F}_t\right)}$$

As from (50), $U_t = e^{-h_S W_t + \left(r - \frac{3}{2}h_S^2\right)t} = Z_t^0 e^{(r-h_S^2)t}$ and $\mathbb{Q}^0$ is defined in (5) by $d\mathbb{Q}^0/d\mathbb{P} = Z_T^0$. We obtain using Bayes Formula again that

$$\mathbb{E}\left(U_T(V_T-K)_+|\mathcal{F}_t\right) = e^{(r-h_S^2)(T-t)} U_t \mathbb{E}^{\mathbb{Q}^0}\left((V_T-K)_+|\mathcal{F}_t\right).$$

As

$$\mathbb{E}\left({U_T}^2|\mathcal{F}_t\right) = e^{-2h_S W_t + (2r-3h_S^2)T + 2h_S^2(T-t)} = U_t^2 e^{(2r-h_S^2)(T-t)}, \tag{71}$$

we get that

$$K_t = e^{-r(T-t)} \frac{\mathbb{E}^{\mathbb{Q}^0}\left((V_T-K)_+|\mathcal{F}_t\right)}{U_t}.$$

But, the process V under the probability $\mathbb{Q}^0$, is a geometric Brownian motion, and we can achieve these decomposition using the Black-Scholes formula. In fact

$$\begin{aligned}\frac{dV_t}{V_t} &= \mu_V V_t dt + \sigma_V V_t(\rho dW_t + \sqrt{1-\rho^2}dW_t^*) \\ &= (\mu_V - \sigma_V h_S \rho)V_t dt + \sigma_V V_t(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t^{*,0}).\end{aligned}$$

The processes W^0 and $W^{0,*}$ are Brownian motion under $\mathbb{Q}^0$ (see Equation (26) for definition). It follows from Black-Scholes formula (56) that

$$K_t = \frac{e^{-r(T-t)}}{U_t} BS(V_t, T-t, K, \mu_V - \sigma_V h_S \rho, \sigma_V)$$

and by formula (57)

$$\begin{aligned}&dBS(V_t, T-t, K, \mu_V - \sigma_V h_S \rho, \sigma_V) \\ &\quad = \sigma_V e^{(\mu_V - \sigma_V h_S \rho)(T-t)} V_t \mathcal{N}(d_1)(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t^{*,0}),\end{aligned}$$

where we used the short notation d_1 for $d_1(V_t, T-t, K, \mu_V - \sigma_V h_S\rho, \sigma_V)$. Using Integration by part formula,

$$
\begin{aligned}
dK_t = &\frac{e^{-r(T-t)}}{U_t} d(BS(V_t, T-t, K, \mu_V - \sigma_V h_S\rho, \sigma_V)) + \\
&BS(V_t, T-t, K, \mu_V - \sigma_V h_S\rho, \sigma_V) d\frac{e^{-r(T-t)}}{U_t} + \\
&d < \frac{e^{-r(T-t)}}{U_t}, BS(V_t, T-t, K, \mu_V - \sigma_V h_S\rho, \sigma_V) > .
\end{aligned}
$$

Using Ito formula,

$$
d\frac{e^{-r(T-t)}}{U_t} = \frac{e^{-r(T-t)}}{U_t}\left(h_S^2 dt + h_S dW_t^0\right)
$$

Thus

$$
d < \frac{e^{-r(T-t)}}{U_t}, BS > = e^{(\mu_V - \sigma_V h_S\rho)(T-t)}\sigma_V\rho h_S \frac{e^{-r(T-t)}}{U_t} V_t\mathcal{N}(d_1)dt
$$

And

$$
\begin{aligned}
dK_t &= \left[h_S^2 K_t + e^{(\mu_V - \sigma_V h_S\rho)(T-t)}\sigma_V\rho h_S \frac{e^{-r(T-t)}}{U_t} V_t\mathcal{N}(d_1)\right] dt + h_S K_t dW_t^0 + \\
&\quad \sigma_V e^{(\mu_V - \sigma_V h_S\rho)(T-t)} \frac{e^{-r(T-t)}}{U_t} V_t\mathcal{N}(d_1)(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t^{*,0}) \\
&= \left(h_S K_t + \rho\sigma_V e^{(\mu_V - \sigma_V h_S\rho)(T-t)} \frac{e^{-r(T-t)}}{U_t} V_t\mathcal{N}(d_1)\right) dW_t^U + \\
&\quad \sqrt{1-\rho^2}\sigma_V e^{(\mu_V - \sigma_V h_S\rho)(T-t)} \frac{e^{-r(T-t)}}{U_t} V_t\mathcal{N}(d_1) dW_t^{*,U}
\end{aligned}
$$

See Equation (52) for definition of W^U and $W^{*,U}$: $W_t^U = W_t^0 + h_S t$ and $W_t^{*,U} = W_t^0$. So we get that

$$
dK_t = \left(h_S K_t + \rho\frac{L_t}{U_t}\right) dW_t^U + \sqrt{1-\rho^2}\frac{L_t}{U_t} dW_t^{*,U} \tag{72}
$$

with

$$
L_t = \sigma_V e^{-r(T-t)+(\mu_V - \sigma_V h_S\rho)(T-t)} V_t\mathcal{N}(d_1). \tag{73}
$$

Going back to the Galtchouk-Kunita-Watanabe (68) of K_t, we are looking for $\Phi^{0,H}$, $\Phi^{1,H}$ and b such that

$$
\begin{aligned}
dK_t &= \Phi_t^{0,H} d\frac{S_t^0}{U_t} + \Phi_t^{1,H} d\frac{S_t}{U_t} + b_t dW_t^{*,U} \\
&= \left(h_S \frac{S_t^0}{U_t}\Phi_t^{0,H} + (h_S + \sigma_S)\frac{S_t}{U_t}\Phi_t^{1,H}\right) dW_t^U + b_t dW_t^{*,U},
\end{aligned}
$$

recall Equations (65) and (66) with $\lambda = 0$. Comparing with equation (72), we obtain that

$$
\begin{aligned}
h_S \frac{S_t^0}{U_t}\Phi_t^{0,H} + (h_S + \sigma_S)\frac{S_t}{U_t}\Phi_t^{1,H} &= h_S K_t + \rho\frac{L_t}{U_t} \\
b_t &= \sqrt{1-\rho^2}\frac{L_t}{U_t}
\end{aligned} \tag{74}
$$

Recall from Theorem 6 that we are looking for $(\Phi^{0,H}, \Phi^{1,H}) \in \mathcal{A}_2^U$. So we impose the self financing condition

$$
\begin{aligned}
\Phi_t^{0,H}\frac{S_t^0}{U_t} + \Phi_t^{1,H}\frac{S_t}{U_t} &= X_0 + \int_0^t \Phi_l^{0,H} d\frac{S_l^0}{U_l} + \int_0^t \Phi_l^{1,H} d\frac{S_l}{U_l} \\
&= X_0 + \int_0^t \left(h_S \frac{S_l^0}{U_l}\Phi_l^{0,H} + (h_S+\sigma_S)\frac{S_l}{U_l}\Phi_l^{1,H} \right) dW_l^U \\
&= X_0 + \int_0^t \left(h_S K_l + \rho \frac{L_l}{U_l} \right) dW_l^U,
\end{aligned}
$$

where we have use Equation (74) to get the last equality. Using equation (74) again, we get that

$$
\begin{aligned}
\Phi_t^{0,H} &= \frac{U_t}{S_t^0}\left[\frac{\sigma_S + h_S}{\sigma_S}\left(X_0 + \int_0^t \left(h_S K_l + \rho\frac{L_l}{U_l}\right) dW_l^U \right) - \frac{1}{\sigma_S}\left(h_S K_t + \rho \frac{L_t}{U_t}\right) \right] \\
\Phi_t^{1,H} &= \frac{U_t}{\sigma_S S_t}\left[\left(h_S K_t + \rho\frac{L_t}{U_t}\right) - h_S\left(X_0 + \int_0^t \left(h_S K_l + \rho\frac{L_l}{U_l}\right) dW_l^U \right) \right]
\end{aligned}
$$

In order to prove that $(\Phi^{0,H}, \Phi^{1,H}) \in \mathcal{A}_2^U$, it remains to prove that $(\Phi^{0,H}, \Phi^{1,H}) \in L\left(\left(\frac{S^0}{U}, \frac{S}{U}\right)\right)$, i.e.

$$
\begin{aligned}
\int_0^T \left(\Phi_t^{0,H}\right)^2 d < \frac{S^0}{U} >_t &= \int_0^T \left(\Phi_t^{0,H}\right)^2 h_S^2 \left(\frac{S_t^0}{U_t}\right)^2 dt < \infty \\
\int_0^T \left(\Phi_t^{1,H}\right)^2 d < \frac{S}{U} >_t &= \int_0^T \left(\Phi_t^{1,H}\right)^2 (h_S+\sigma_S)^2 \left(\frac{S_t}{U_t}\right)^2 dt < \infty
\end{aligned}
$$

This holds true because $\Phi_t^{0,H}$, $\Phi_t^{1,H}$, S_t^0, S_t and U_t are continuous on $[0,T]$.

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